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Dynamic System Stability

DEEM Trainings on March 2019

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Sources:

R. Dorf, R. Bishop. Modern Control Systems. Pearson Prentice Hall 2008. ISBN 10: 0-13-245192-1
K.J. Åström, R.M. Murray. Feedback Systems. Princeton University Press. http://www.cds.Caltech.edu/~murray/FBS

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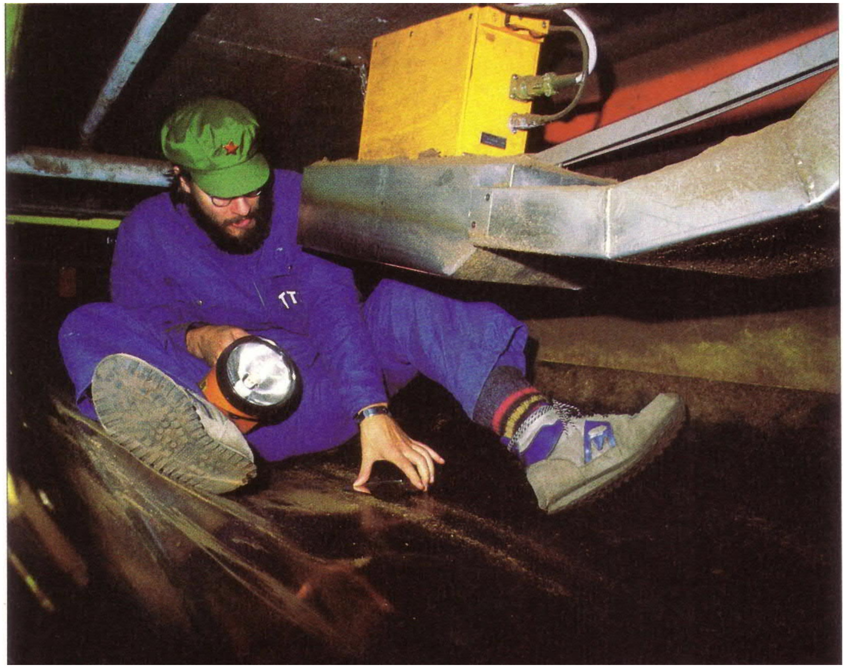
Introduction

Yrjö Majanne

Position: Research manager

Special expertise:

- Power generation and control of power systems
  - Control and Instrumentation
  - Dynamic modelling
  - Process optimization
  - Environmental systems, efficiency, LCA,
- Industrial automation
  - Automation systems (normal & safety related)
  - Field instrumentation
- City energy systems
  - Distributed generation
  - Energy intelligent buildings
  - IoT



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Outline

- Dynamic modelling of physical systems
- Feedback systems
- Transfer function in frequency domain and stability

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Dynamic Modelling of Physical Systems

- Static and dynamic systems
- Differential equations
- Laplace transformation
- Transfer functions

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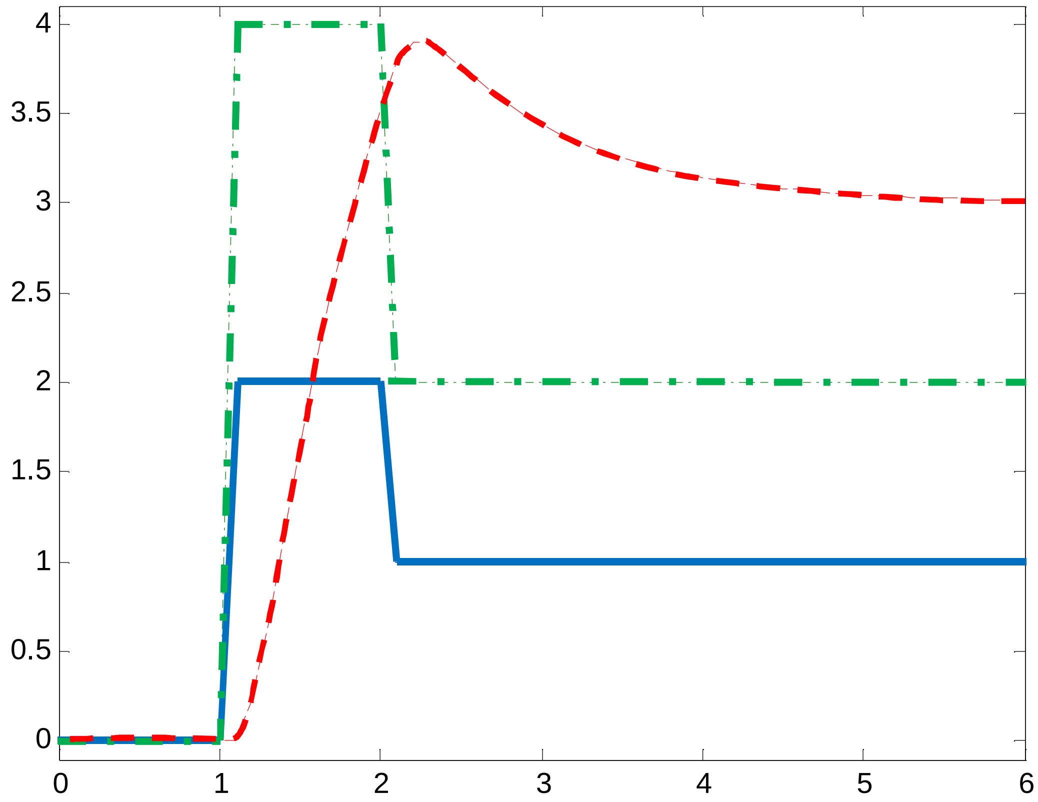
Dynamic Modelling of Physical Systems
Static and Dynamic Systems

- Step input (continuous blue line)
- Static step response (dash dot green line)
- Dynamic step response (dashed red line)

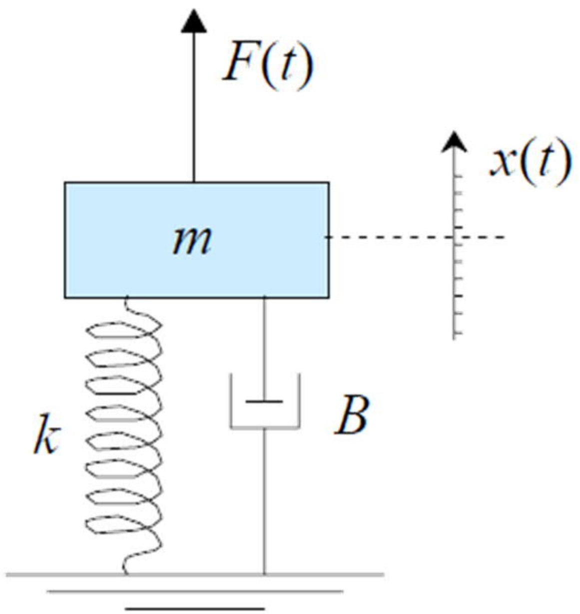
The system is defined **dynamic** when the current state of the system is a function of its previous states and inputs (the system has memory and inertia).



Dynamic Modelling of Physical Systems
Static and Dynamic Systems

- E.g. the effect of the external force *F*(t) to the position of mass *x*(t) is derived from the force balance (*m* = mass, *k* = spring constant, *B* = damping constant)

$$m\frac{d^2x(t)}{dt^2}+B\frac{dx(t)}{dt}+kx(t)=F(t)$$



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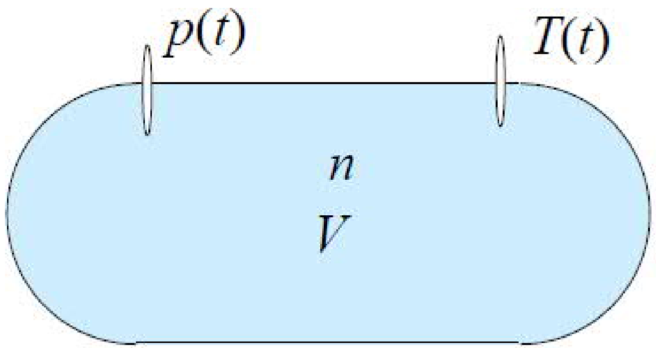
Dynamic Modelling of Physical Systems
Static and Dynamic Systems

- The state of a **static system** does not depend on the previous values of the states and inputs (no memory, no inertia).
- E.g. effect of temperature *T* to pressure *p* in an insulated closed tank
  - Derived from the ideal gas state equation (n = number of moles, V = tank volume, R = universal gas constant)

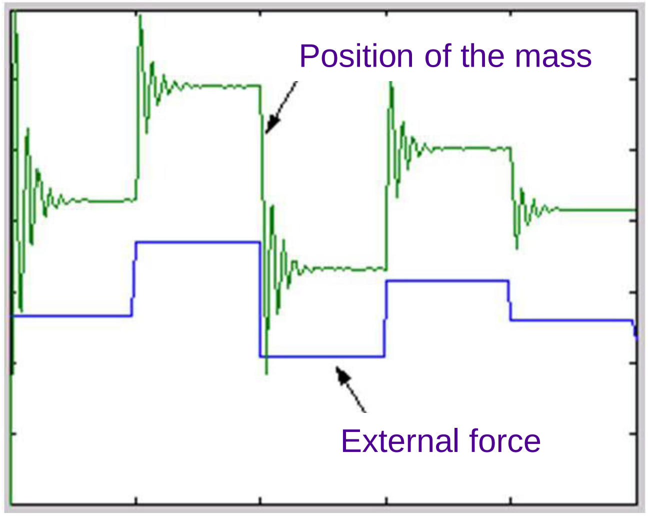
$$p(t)V = nRT(t)$$



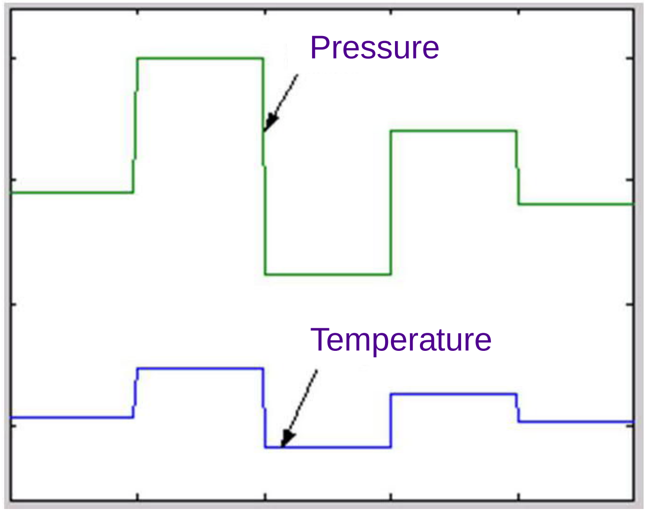
Dynamic Modelling of Physical Systems
Static and Dynamic Systems

- Simulation results where in the mechanical system the external force F and in the gas tank system the temperature T are changed stepwise.

Dynamic mass position



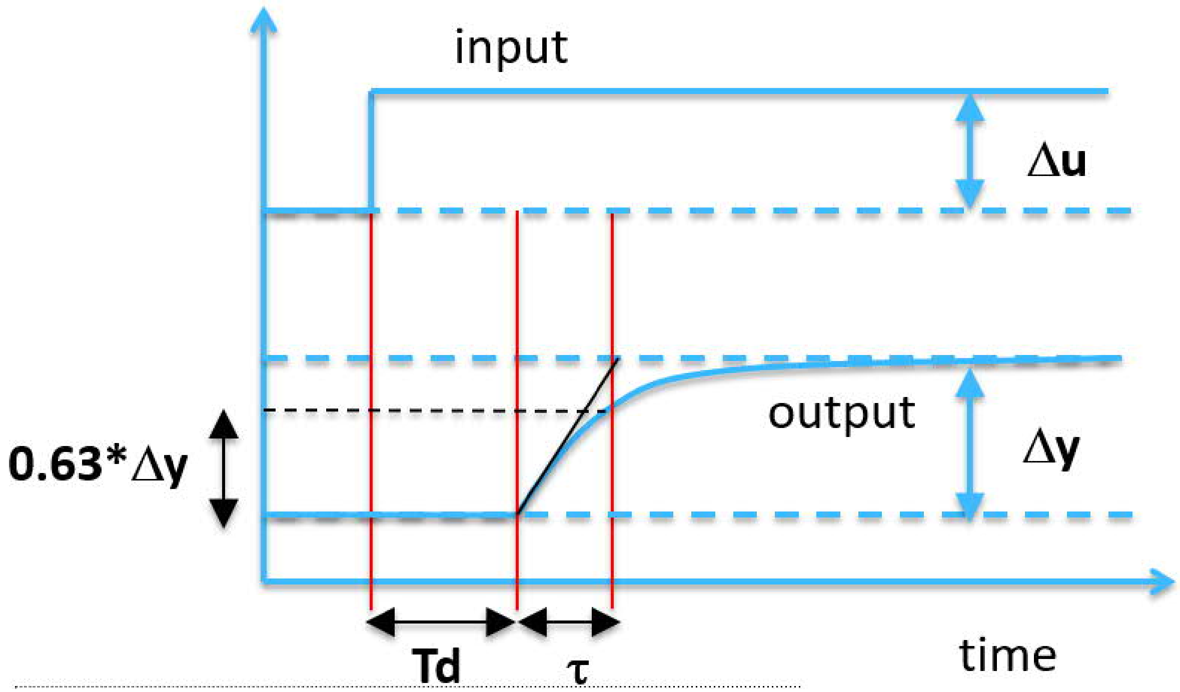
Static gas temperature - pressure relation



Dynamic Modelling of Physical Systems
Static and Dynamic Systems

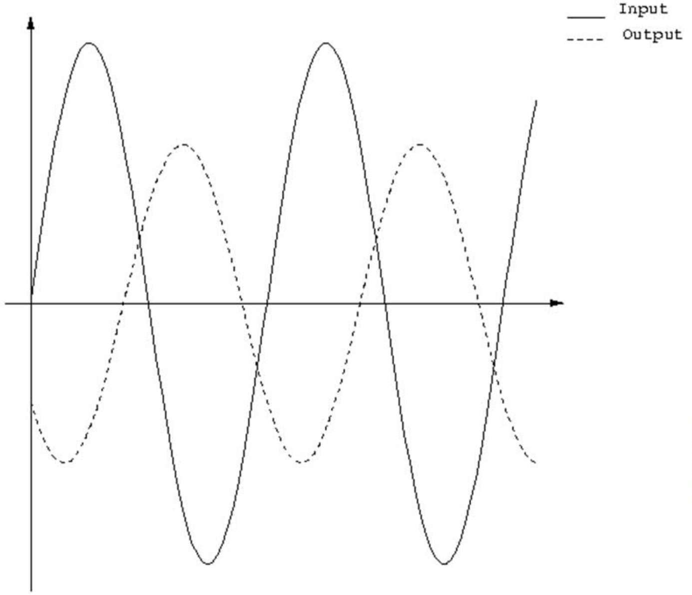
Parameters of the dynamic system

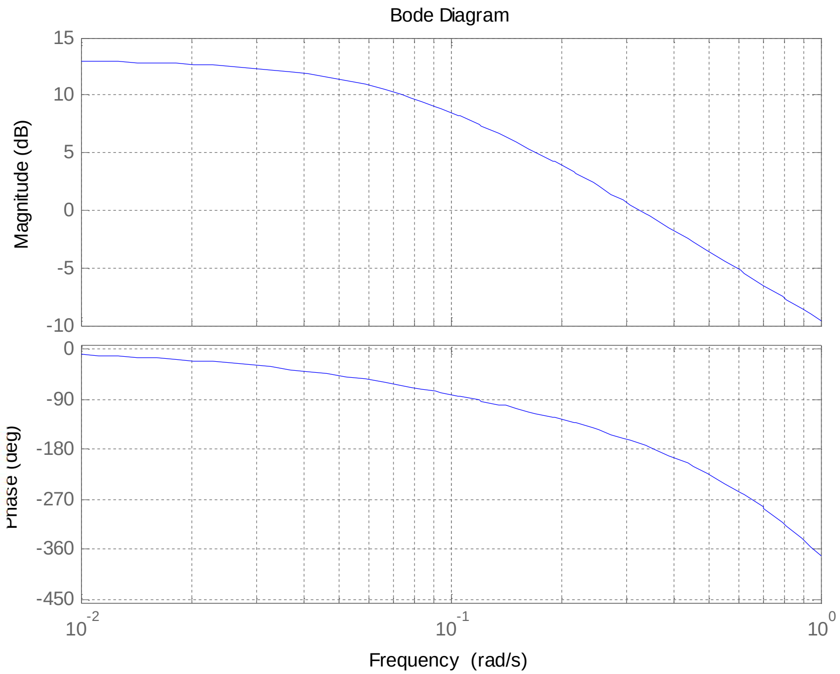
- Gain, $\Delta y/\Delta u$
- Time constant, τ
- Delay, T_d
  - Input delay
  - Output delay



Dynamic Modelling of Physical Systems
Static and Dynamic Systems

- Frequency response of a dynamic system





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Dynamic Modelling of Physical Systems
Static and Dynamic Systems

Complexity of dynamic systems

- System consists of both short and long time constants (stiff system)
- Interactions of multivariate systems
- Feedback dominates the system behaviour
- Nonlinear dynamics
- Irreversible processes (history depencancy)
- Self-organizing and chaotic systems
- Adaptivity (time variant systems)
- Non-intuitivity – cause and effect are far from each other either in time scale or in location.
- Opposite short range and long range effects (non-minimum phase systems)
- Delays

Dynamic Modelling of Physical Systems
Differential Equations

There are two ways of describing differential equations:

- Directly relate inputs *u* to outputs *y* in one differential equation

$$g\left(y^{(n)}(t), y^{(n-1)}(t), \dots, y(t), u^{(m)}(t), u^{(m-1)}(t), \dots, u(t)\right) = 0$$

- where $y^{(k)}(t) = \dfrac{d^k}{dt^k} y(t)$

- and $g(\cdot, \cdot, \dots, \cdot)$ is a nonlinear vector valued function

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Dynamic Modelling of Physical Systems
Differential Equations

- Write the differential equations as a system of first order differential equations by introducing a number of internal variables (state-space system)

Internal state variables

$$x(t)=\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Inputs

$$u(t)=\begin{bmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

Outputs

$$y(t)=\begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

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Dynamic Modelling of Physical Systems
Differential Equations

State Equations

$$\dot{x}(t) = f\left(x(t), u(t)\right) \qquad f(x,u) = \begin{bmatrix} f_1(x,u) \\ \vdots \\ f_n(x,u) \end{bmatrix} \qquad \begin{aligned} \dot{x}_1(t) &= f_1\left(x_1,\ldots,x_n,u_1,\ldots,u_m\right) \\ &\vdots \\ \dot{x}_n(t) &= f_n\left(x_1,\ldots,x_n,u_1,\ldots,u_m\right) \end{aligned}$$

Output or Measurement Equations

$$y(t) = h\left(x(t), u(t)\right) \qquad h(x,u) = \begin{bmatrix} h_1(x,u) \\ \vdots \\ h_p(x,u) \end{bmatrix} \qquad \begin{aligned} y_1(t) &= h_1\left(x_1,\ldots,x_n,u_1,\ldots,u_m\right) \\ &\vdots \\ y_p(t) &= h_p\left(x_1,\ldots,x_n,u_1,\ldots,u_m\right) \end{aligned}$$

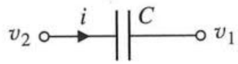
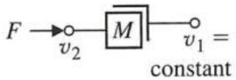
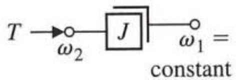
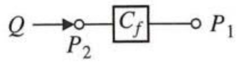
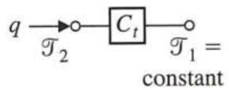
Dynamic Modelling of Physical Systems
Differential Equations

- The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process.
- Balance equations

  E.g. Energy balance $\frac{dE}{dt} = P_{in} - P_{out}$ E = Energy [J], P = power [W]

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Dynamic Modelling of Physical Systems
Differential Equations

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}Cv_{21}{}^2$	
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}Mv_2{}^2$	
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J\omega_2{}^2$	
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_fP_{21}{}^2$	
	Thermal capacitance	$q = C_t\frac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	

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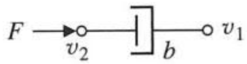
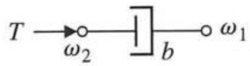
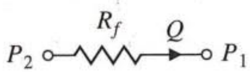
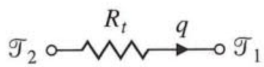
Dynamic Modelling of Physical Systems
Differential Equations

Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Energy dissipators	Electrical resistance	$i = \frac{1}{R} v_{21}$	$\mathcal{P} = \frac{1}{R} {v_{21}}^2$	
	Translational damper	$F = b v_{21}$	$\mathcal{P} = b {v_{21}}^2$	
	Rotational damper	$T = b \omega_{21}$	$\mathcal{P} = b {\omega_{21}}^2$	
	Fluid resistance	$Q = \frac{1}{R_f} P_{21}$	$\mathcal{P} = \frac{1}{R_f} {P_{21}}^2$	
	Thermal resistance	$q = \frac{1}{R_t} \mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t} \mathcal{T}_{21}$	

Dynamic Modelling of Physical Systems
Laplace Transformation

- The Laplace transform method converts the more difficult differential equations to relatively easily solved algebraic equations. The time-response solution is obtained by the following operations:

1. Obtain the linearized differential equations.
2. Obtain the Laplace transformation of the differential equations.
3. Solve the resulting algebraic equation for the transform of the variable of interest.

- The Laplace variable *s* can be considered to be the differential and integral operators so that

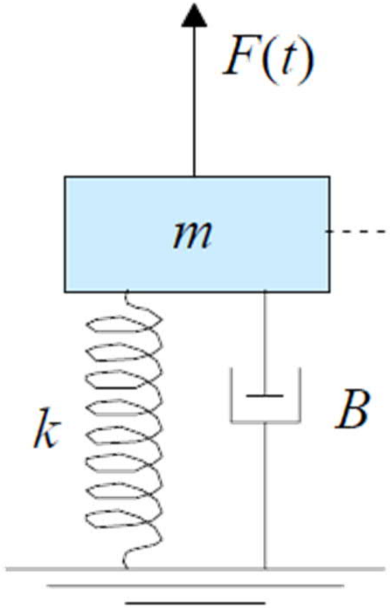
$$s = \frac{d}{dt} \qquad \frac{1}{s} = \int_0^t dt$$

Dynamic Modelling of Physical Systems
Laplace Transformation

- Example: Spring – mass – damper system

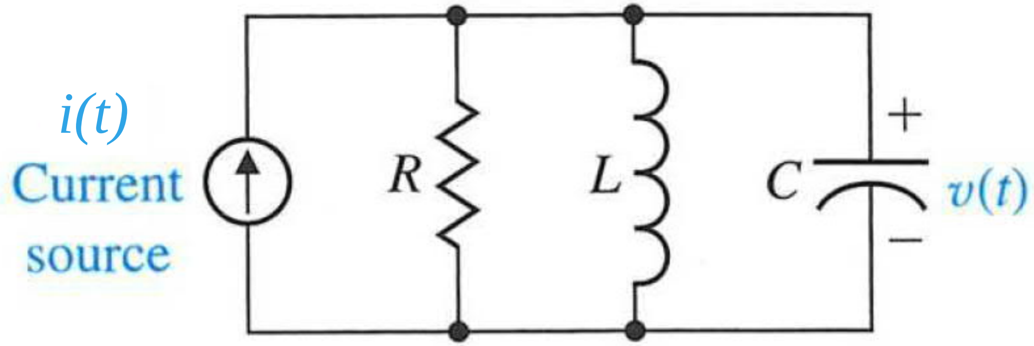
$$m\frac{d^2x(t)}{dt^2}+B\frac{dx(t)}{dt}+kx(t)=F(t)$$

$$L\left(F(t)\right)=F(s)=ms^2X(s)+BsX(s)+kx(s)$$



Dynamic Modelling of Physical Systems
Laplace Transformation

- Example: Modelling of the electrical RLC circuit by utilizing Kirchhoff's current law.



$$\frac{v(t)}{R}+C\frac{dv(t)}{dt}+\frac{1}{L}\int_0^t v(t)\,dt=i(t)$$

$$\mathrm{L}\left(i(t)\right)=I(s)=\frac{V(s)}{R}+CsV(s)+\frac{1}{Ls}V(s)$$

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Dynamic Modelling of Physical Systems
Transfer Function

- The transfer function of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero.
- A transfer function may be defined only for a linear, stationary (constant parameter) system.
- Furthermore, a transfer function is an input—output description of the behavior of a system.
  - Thus, the transfer function description does not include any information concerning the internal structure of the system and its behavior.

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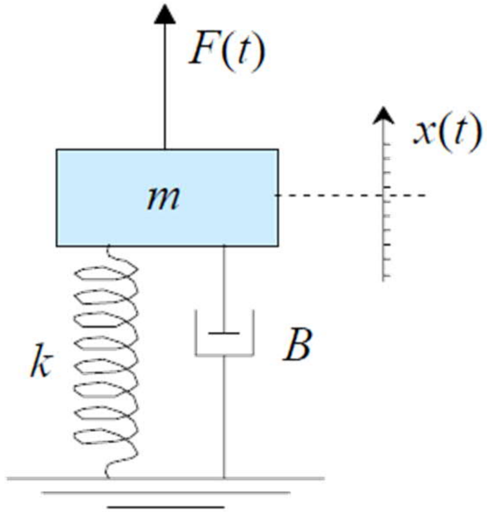
Dynamic Modelling of Physical Systems
Transfer Function

- Example: Spring – mass – damper system



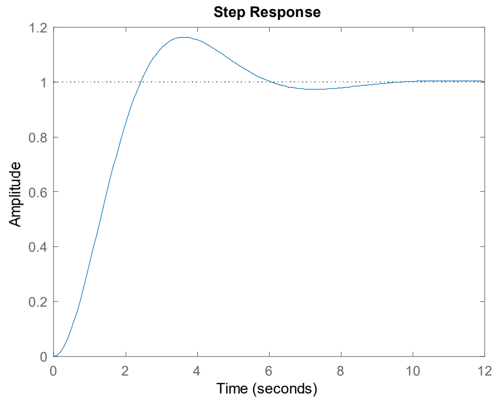
$$m\frac{d^2x(t)}{dt^2}+B\frac{dx(t)}{dt}+kx(t)=F(t)$$

$$F(s)=ms^2X(s)+BsX(s)+kX(s)$$

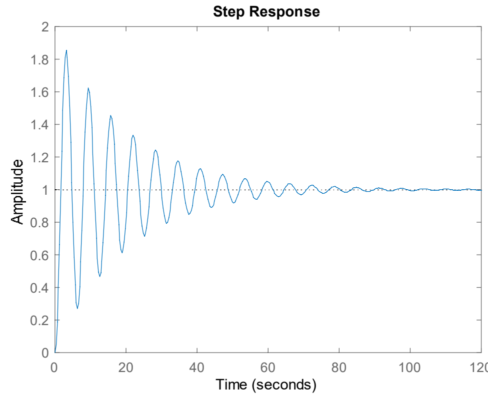
- Transfer function

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{ms^2+Bs+k}$$

B = 1



B = 0.1



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Dynamic Modelling of Physical Systems
Transfer Function

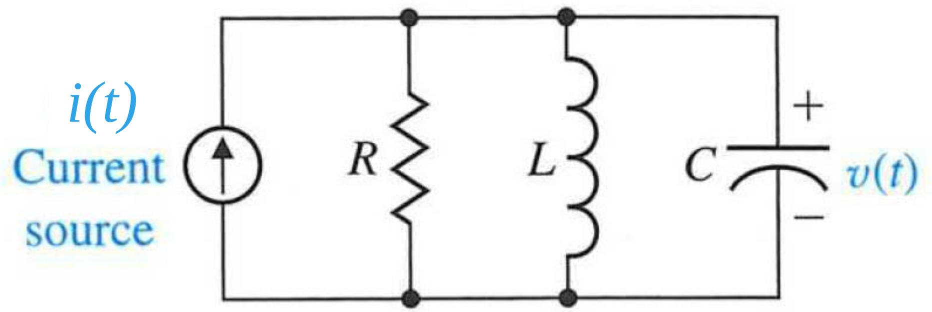
- Example: Modelling of the electrical RLC circuit

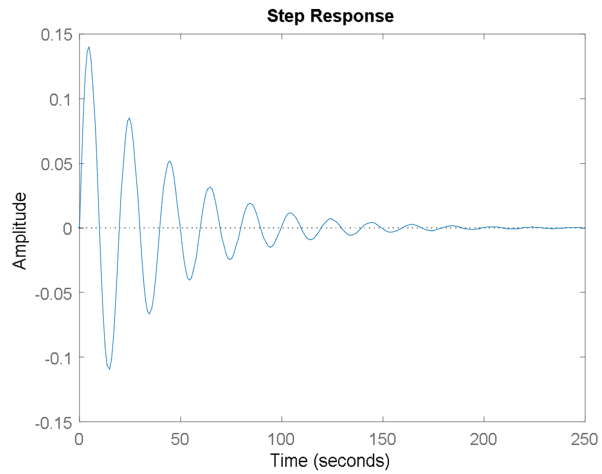
$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = i(t)$$

$$L(i(t)) = I(s) = \frac{V(s)}{R} + CsV(s) + \frac{1}{Ls}V(s)$$

- Transfer function

$$G_{RCL}(s) = \frac{V(s)}{I(s)} = \frac{Ls}{RCLs^2 + Ls + R}$$





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Feedback Systems

- Feedback structure
- Feedback properties

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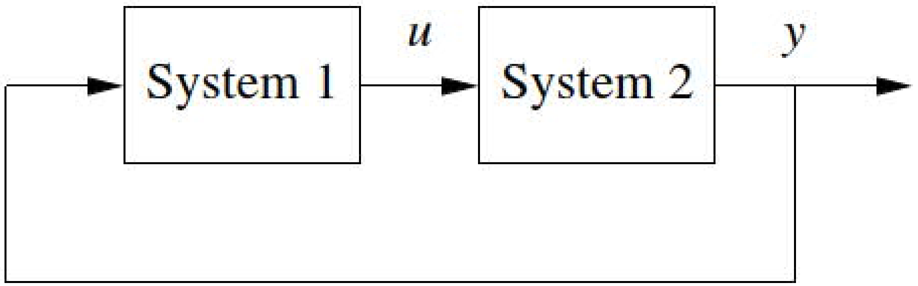
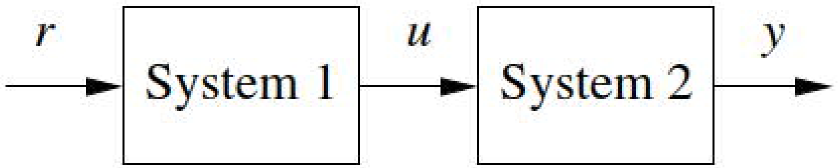
Feedback System

- The term feedback refers to a situation in which two (or more) dynamical systems are connected together such that each system influences the other and their dynamics are thus strongly coupled.
- Simple causal reasoning about a feedback system is difficult because the first system influences the second and the second system influences the first, leading to a circular argument.



(a) Closed loop



(b) Open loop

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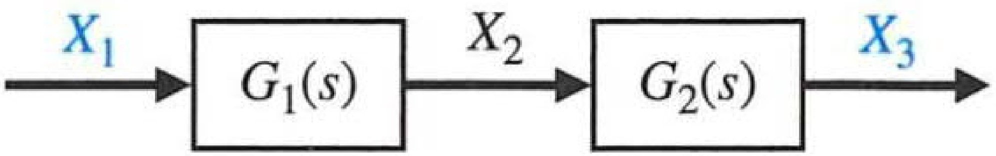
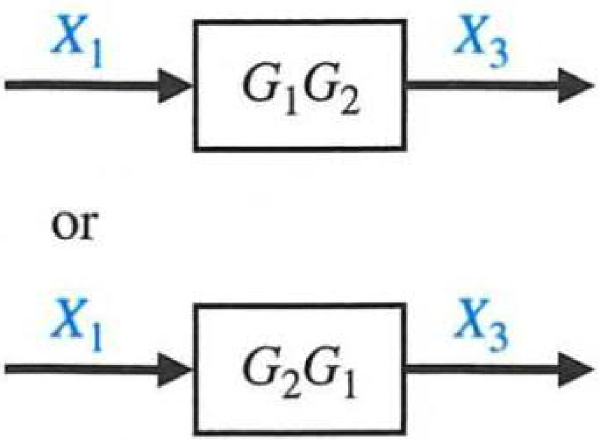
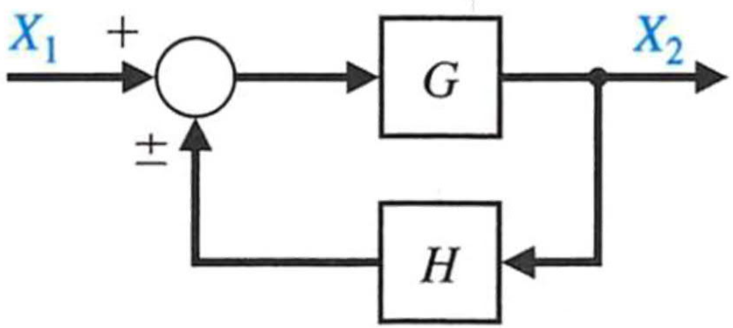
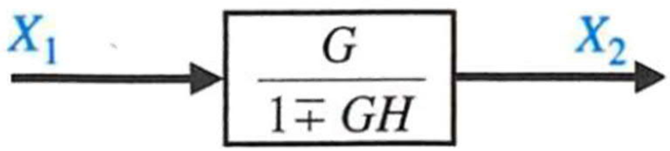
Feedback System









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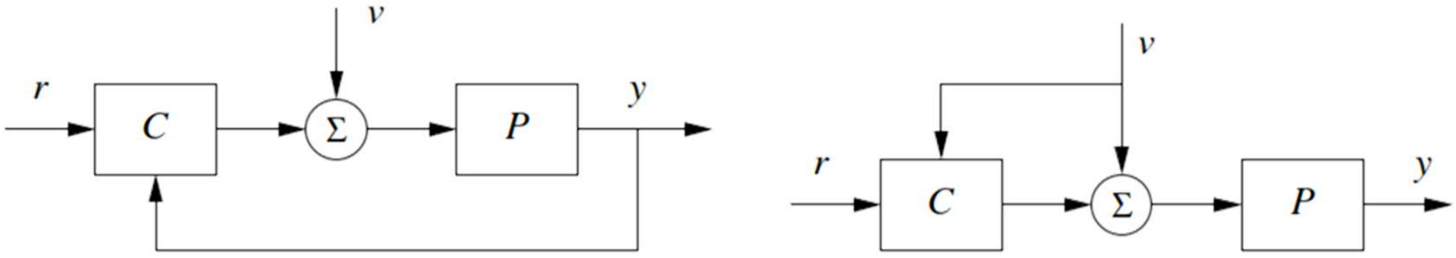
Feedback System

- Feedback has many interesting properties that can be exploited in designing systems.
  - make a system resilient toward external influences
  - create linear behavior out of nonlinear components, a common approach e.g. in electronics
  - allows a system to be insensitive both to external disturbances *v* and to variations in its individual elements.
- Feedback has potential disadvantages as well
  - create dynamic instabilities in a system
  - causing oscillations or even runaway behavior
  - introduce unwanted sensor noise into the system requiring careful filtering of signals



(a) Feedback system (b) Feedforward system

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Feedback System

Table 1.1: Properties of feedback and feedforward

Feedback	Feedforward
Closed loop	Open loop
Acts on deviations	Acts on plans
Robust to model uncertainty	Sensitive to model uncertainty
Risk for instability	No risk for instability

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Feedback System
Properties

Robustness to Uncertainty

- supply a corrective action to partially compensate for the effect of disturbances
  - load
  - other
- provide robustness to variations in the process dynamics
- provides robust performance in the presence of uncertain dynamics

Design of Dynamics

- Another use of feedback is to change the dynamics of a system.
  - systems that are unstable can be stabilized,
  - systems that are sluggish can be made responsive
  - systems that have drifting operating points can be held constant.

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Feedback System
Properties

- Control theory provides a rich collection of techniques to analyze the stability and dynamic response of complex systems and to place bounds on the behavior of such systems by analyzing the gains of linear and nonlinear operators that describe their components.

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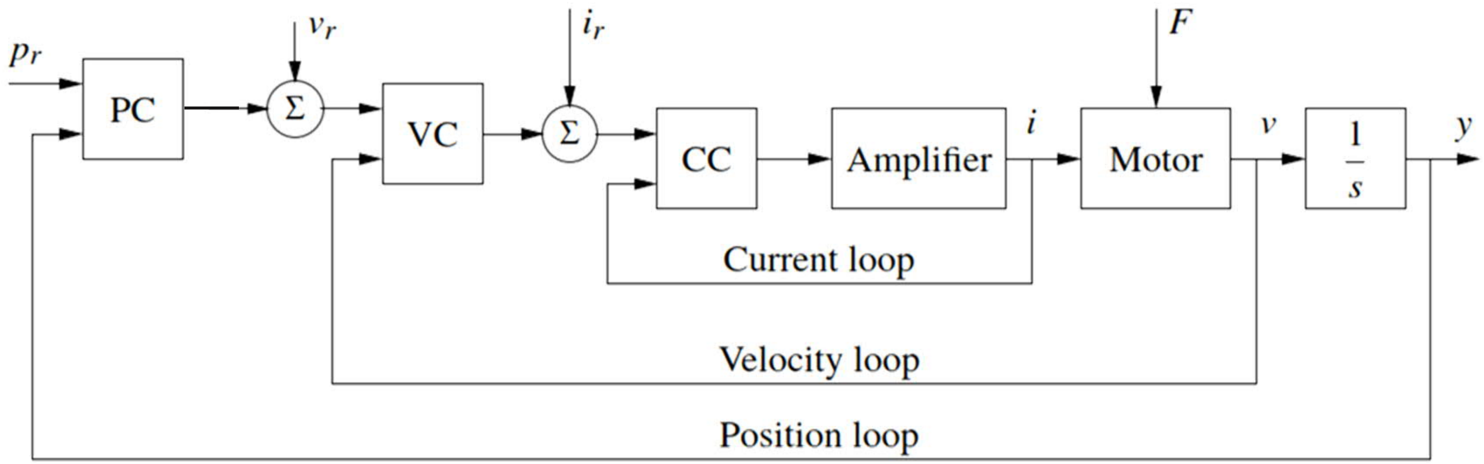
Feedback System
Properties

Creating Modularity

- Feedback can be used to create modularity and shape well-defined relations between inputs and outputs in a structured hierarchical manner. A modular system is one in which individual components can be replaced without having to modify the entire system.
- By using feedback, it is possible to allow components to maintain their input/output properties in a manner that is robust to changes in its interconnections.



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Feedback System
Properties

Challenges of Feedback

- the possibility of instability if the system is not designed properly
- we must design the system not only to be stable under nominal conditions but also to remain stable under all possible perturbations of the dynamics
- feedback inherently couples different parts of a system. One common problem is that feedback often injects measurement noise into the system.
- Measurements must be carefully filtered so that the actuation and process dynamics do not respond to them, while at the same time ensuring that the measurement signal from the sensor is properly coupled into the closed loop dynamics (so that the proper levels of performance are achieved).

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Feedback System
Properties

Challenges of Feedback …

- Another potential drawback of control is the complexity of embedding a control system in a product.
- While the cost of sensing, computation, and actuation has decreased dramatically in the past few decades, the fact remains that control systems are often complicated, and hence one must carefully balance the costs and benefits.

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Transfer function in frequency domain

- Frequency response vs. time response
- Dynamic characteristics in frequency domain
- The role of the poles and the zeros to the frequency response

Transfer function in frequency domain

- Frequency response can be calculated from transfer function by substituting the Laplace operator *s* with $j\omega$, $s = j\omega$
- G(jω) is a complex number function and it can be expressed in a complex plane

$$G(j\omega) = R(j\omega) + jX(j\omega)$$

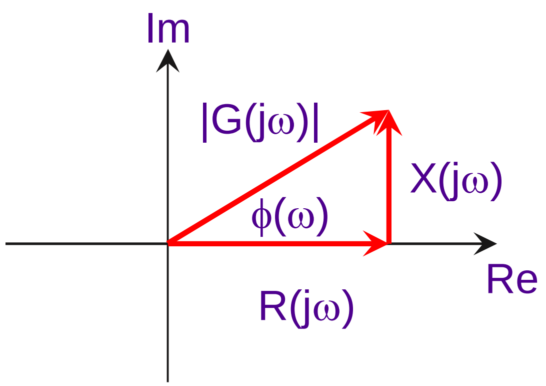
where $R(j\omega) = \mathrm{Re}\big(G(j\omega)\big)$ and $X(j\omega) = \mathrm{Im}\big(G(j\omega)\big)$

- Alternatively, the transfer function can be expressed in polar coordinates

$$G(j\omega) = \left|G(j\omega)\right| e^{j\phi(\omega)}$$

where

- $\phi(\omega) = \tan^{-1} X(\omega)/R(\omega)$ and $\left|G(j\omega)\right| = \sqrt{\left(R(\omega)\right)^2 + \left(X(\omega)\right)^2}$



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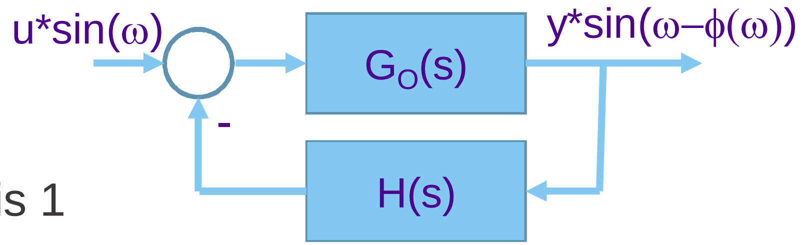
Transfer function in frequency domain



Why frequency response is important

- The important things for stability question are
  - what is the phase lag of the system when the system gain is 1
  - what is the system gain when the phase lag is -180° deg.
- If the phase lag of the open system with a certain frequency is -180° deg. and at this same frequency open system gain $|G_O|$ is ≥ 1, it follows that
  - when the open system is closed with the negative feed back loop with unit gain (H(s) = 1)
  - input of the system is sine wave with this frequency
  - the feed back signal is summed with the input signal in the same phase resulting increased oscillation
    - equal phase lag in input and feed back signal is a result of the negative feed back connection (feedback signal multiplied with -1)

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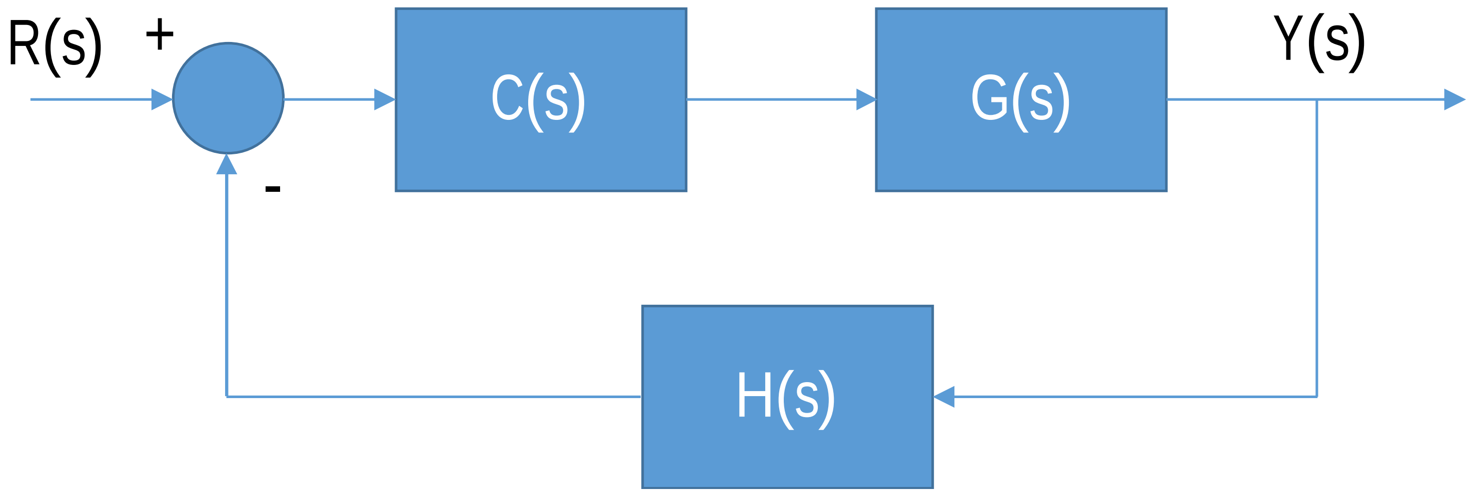
Transfer function in frequency domain



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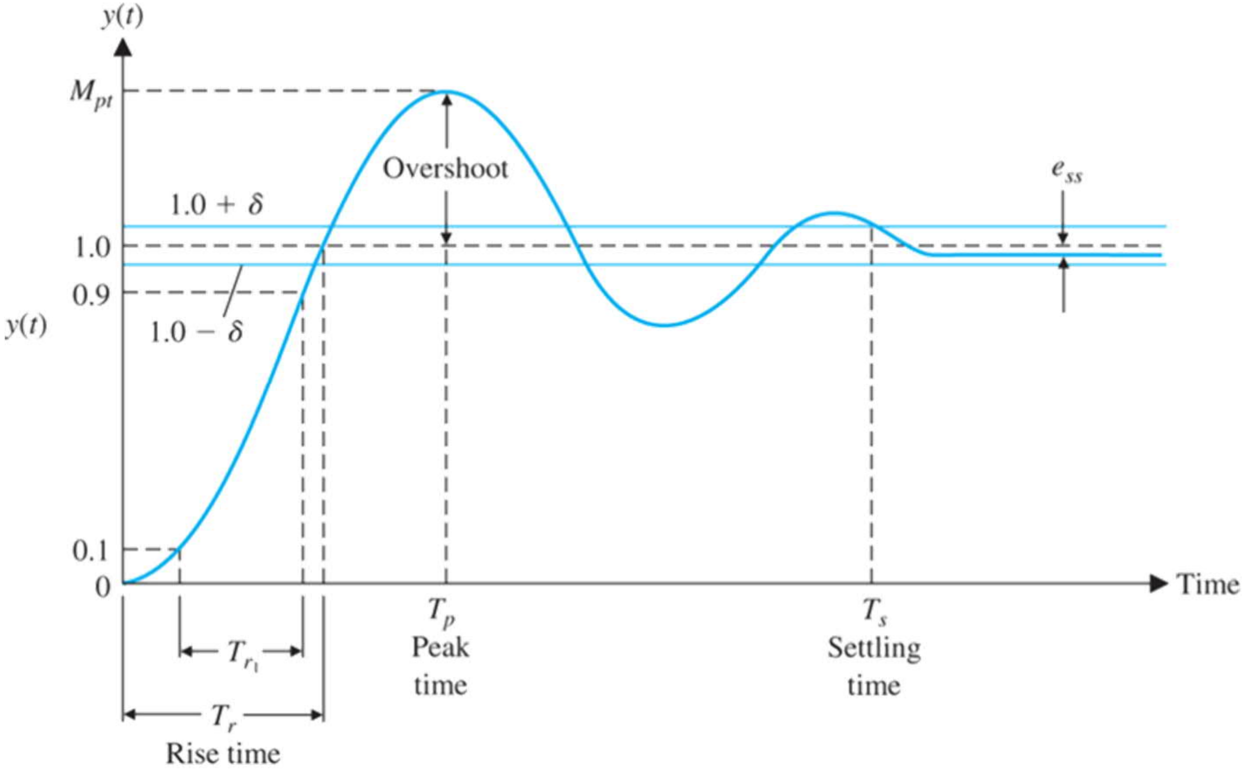
Transfer function in frequency domain
Dynamic characteristics in frequency domain

- In time domain:



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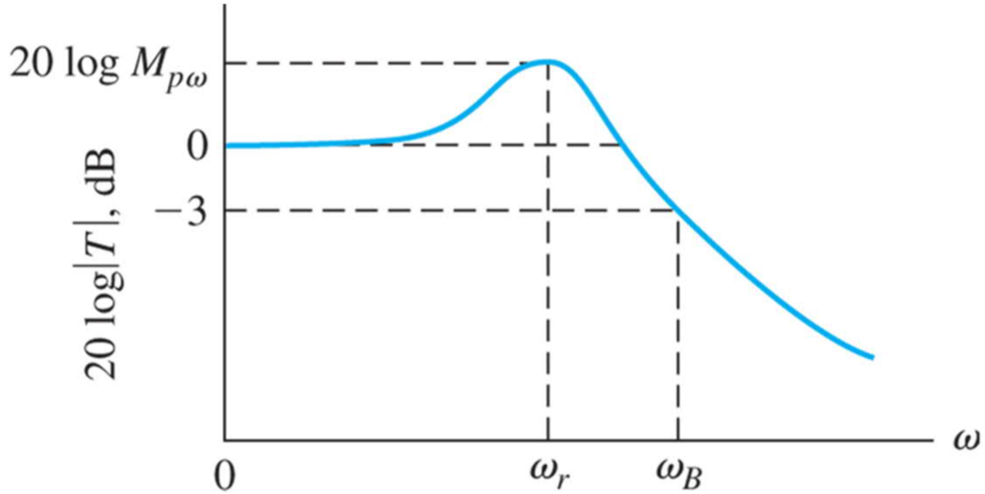
Transfer function in frequency domain
Dynamic characteristics in frequency domain

- How about in frequency domain?
  - Transfer function of an ideal second order system is

$$T(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

ω_n = nominal frequency rad^{-1}
ζ = damping coefficient

  - Transfer response of this system is depicted ->



- Definitions:
  - the biggest gain of the frequency response, $M_{p\omega}$, is reached at the resonance frequency ω_r
  - system bandwidth is the frequency band from 0 to ω_B , where the system gain is damped to -3 dB

ω_r = resonance frequency, rad^{-1}
$M_{p\omega}$ = peak resonance gain, dB
ω_B = system band width

Transfer function in frequency domain
Dynamic characteristics in frequency domain

Links between time and frequency domains

- Based on the comparison of the characteristics of the time domain and frequency domain responses of the second order system
  - in many real systems there exist a so called dominating pair of poles, which means that the dominating dynamics of the system is based on this one pair of poles locating nearest to the origin of the complex plain (poles having the slowest dynamics)
    - the effect of faster poles and pairs of poles can be neglected

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Transfer function in frequency domain
Dynamic characteristics in frequency domain

Links between time and frequency domains

- If the dynamics of the system are defined by the dominating pair of poles (2nd order system), then we can make the following approximations
  - the damping coefficient of the second order system is connected with the peak resonance gain $M_{p\omega}$
  - increased peak resonance gain implies with percentual overshoot of the time response
  - system bandwidth correlates with the set point tracking capability of the system
  - when the bandwidth and resonance frequency are increased, the rise time of the system is decreased
  - steady state error of the system depends on the low frequency gain of the system

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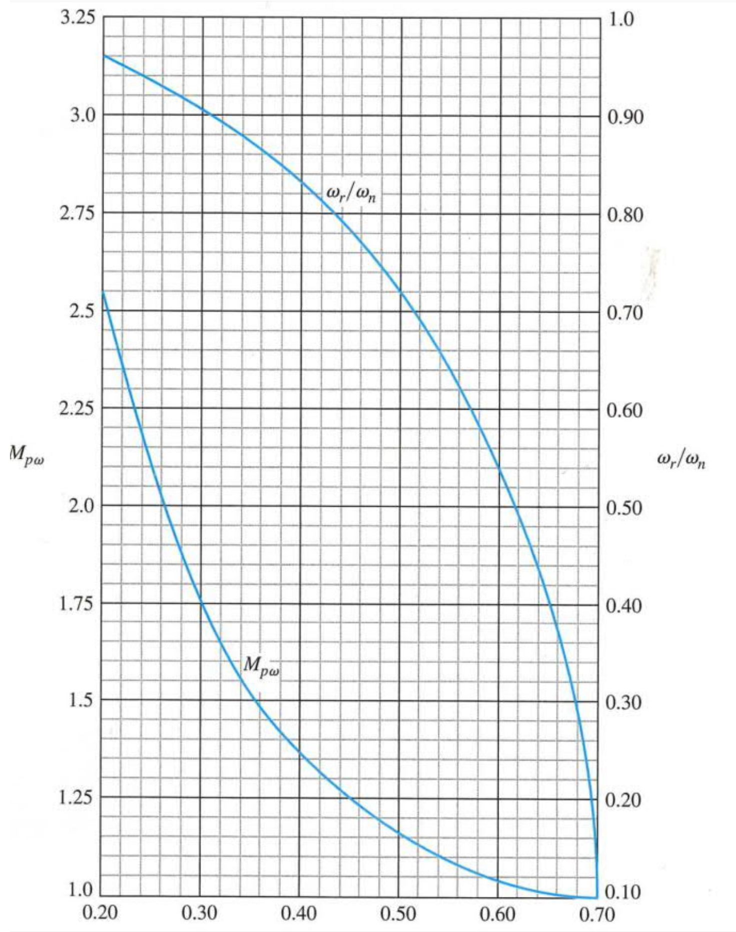
Transfer function in frequency domain
Dynamic characteristics in frequency domain

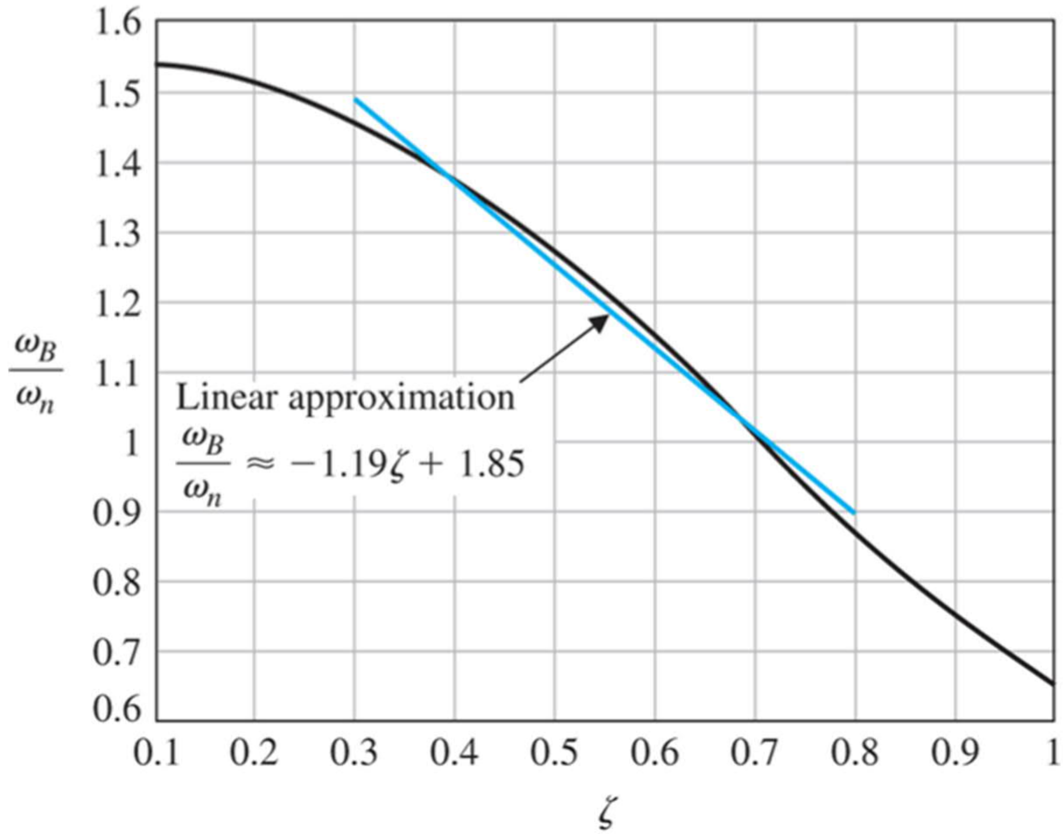
- The peak resonance gain $M_{p\omega}$ and the normalized resonance frequency ω_r/ω_n as functions of the damping coefficient ζ



Transfer function in frequency domain

Dynamic characteristics in frequency domain

- Normalized bandwidth ω_B/ω_n as a function of the damping coefficient ζ .
- The correlation can be approximated with the linear function: $\omega_B/\omega_n = -1.19\,\zeta + 1.85$, when the values of damping coefficient is between $0.3 \le \zeta \le 0.8$



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Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- Each pole and zero effect on the system gain and phase behavior as a function of system frequency

$$G(s)=\frac{(s+z_1)(s+z_2)}{(s+p_1)(s^2+bs+q)}$$

$s = j\omega$, and let's look the system term by term, e.g. $(s+p_1)$ -> $(j\omega+p_1)$

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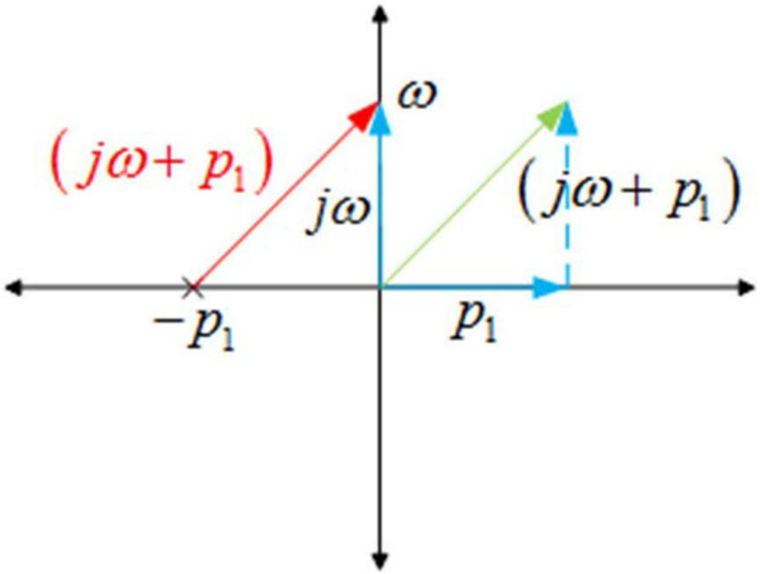
Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

Real pole ($s+p_1$)

- Let's look first the case of real pole describing stable dynamics (($j\omega+p_1$), $p_1 > 0$ => the pole is located at the left half plane of the complex plane, $j\omega = -p_1$)
- The term can be thought also as a vector in the complex plane (real part p_1 and imaginary part $j\omega$). and the term ($j\omega+p_1$) is the sum (green vector) of the two complex vectors $j\omega$ and p_1 (blue vectors)).
- If the sum vector is transferred to start at the corresponding pole –p1 (red vector), the tip of the vector is located at the imaginary axis at the point $j\omega$.
- The length of the vector is related with the gain of the transfer function and vector angle is related with the phase difference. Both vector length and the angle are functions of frequency ω.



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Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

An illustrative example

- When analyzing the effect of the pole to the system gain, it must be remembered that the pole is located at the denominator of the transfer function. A simplified transfer function

$$G(s) = \frac{1}{(s+p)}$$

- having one pole located on the negative real axis at the point –p. Let's mark (s+p) = P(s). Thus this term can be expressed as a function of gain and phase

$$P(s) = |P(s)| \angle P(s)$$

- According to calculation rules of complex numbers, the inverse of P(s), $\frac{1}{P(s)}$ is a complex number, whose length is $\frac{1}{|P(s)|}$ and angle $-\angle P(s)$

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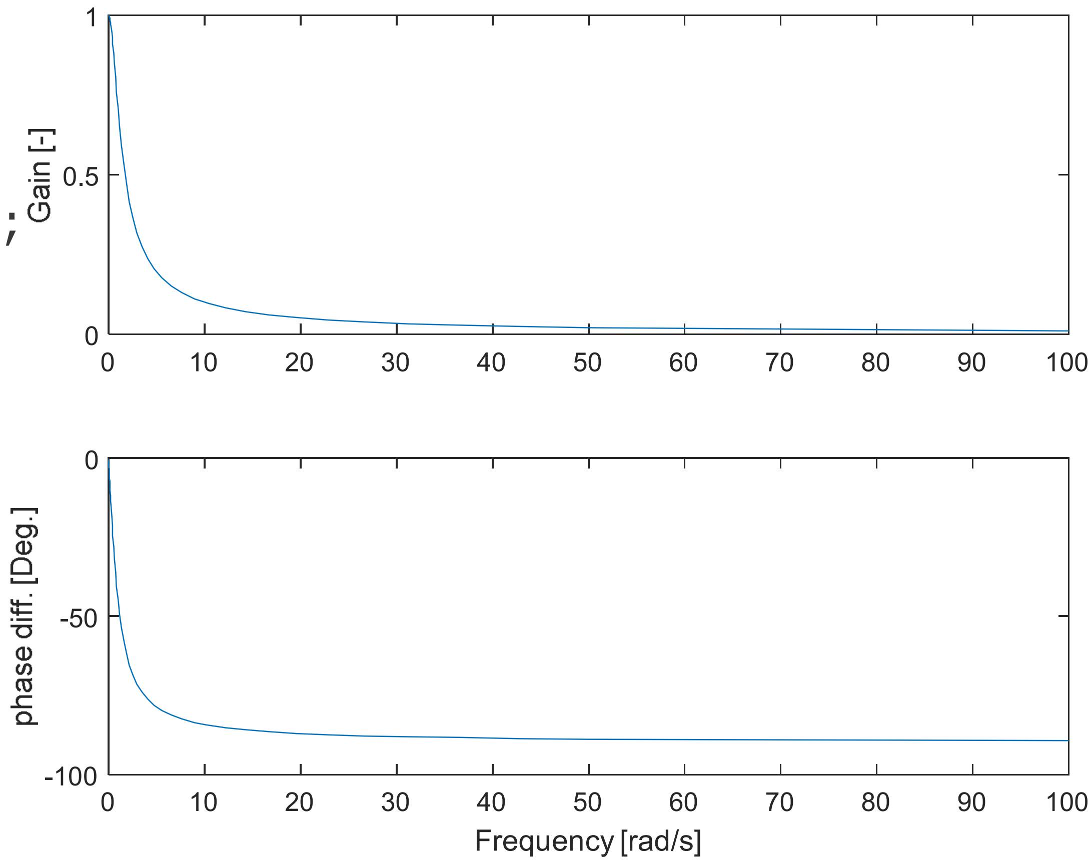
Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

Matlab script:

```
s=tf('s')
G=1/(1+s)
[mag,phase,wout] = bode(G);

figure(1)
subplot(211)
plot(wout,mag(:))
subplot(212)
plot(wout,phase(:))
```



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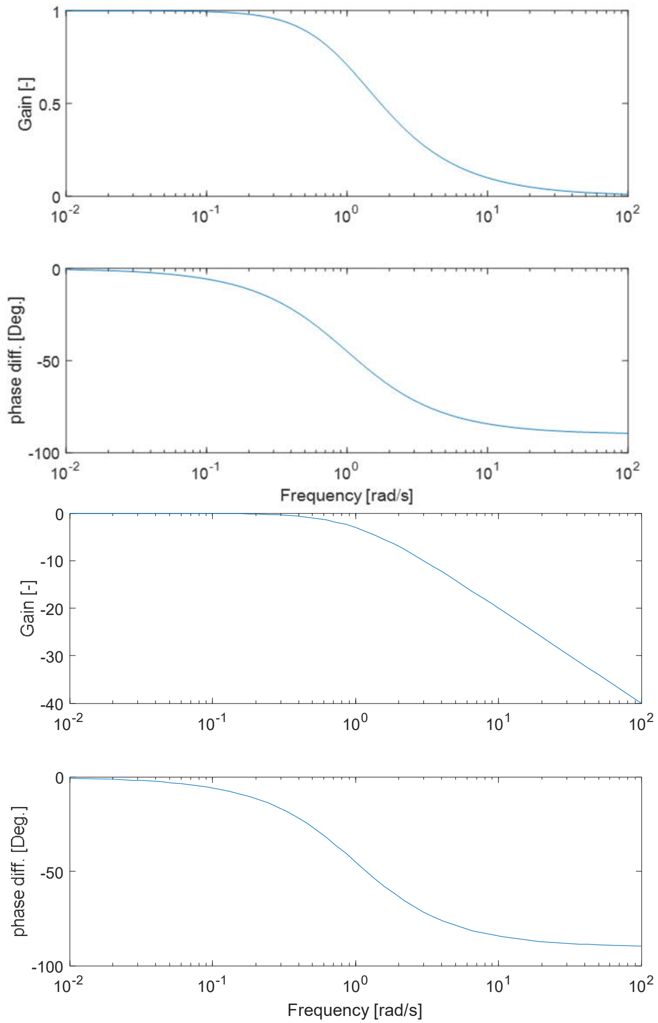
Transfer function in frequency domain

The role of the poles and the zeros to the frequency response

- Figure shows that with increasing frequency ω, the length of the vector is increasing and its invers approaches zero. Respectively, at low frequencies the vector angle is near zero but approaches +90 Deg. with increasing frequency. Because the term is a pole, its effect to the phase of the transfer function is negative (lagging) and it approaches -90 Deg.
- In frequency response plots the frequency axis is logarithmic, which means that in plots distance between 1 – 10 is equal with distance between 10 – 100, etc. Also gain is typically expressed as desibels, [dB]
- Gain = $20 \log_{10}|G(j\omega)|$



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Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

Complex pair of poles

- The pair of poles of the transfer function is defined by the term ($s^2 + ps + q$). Thus the poles are (denominator = 0 and the whole transfer function is infinite)
- $s_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$, thus $s_1 = -\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}$ and $s_2 = -\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}$
- When the roots of the polynome $\left(s^2 + ps + q\right)$ are known, it can be rewritten with its factors $\left(s - s_1\right)\left(s - s_2\right)$
- If the roots of the polynome are real, thus $\left(\frac{p^2}{4} - q\right) \geq 0$, the poles of the system are located on the real axis.

Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- If the roots of the polynome are complex, thus $\left(\frac{p^2}{4}-q\right)\leq 0$, the poles of the system are

  located on on the complex plane, and their real parts are $-\frac{p}{2}$, and imaginary parts for

  the first pole is $j\sqrt{q-\frac{p^2}{4}}$ and for the second pole $-j\sqrt{q-\frac{p^2}{4}}$

Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- Lets illustrate this by plotting the case where the coefficients p and q are chosen so that the system has complex poles. Let's assume that p = 2 and q = 2. The complex pair of poles is defined from the term $\left(s^2+2s+2\right)$, and the poles are $p_{1,2}=-1\pm j$
- The term $\left(s^2+2s+2\right)$ can be written now with its factors

$$\left(s^2+2s+2\right)=\left(s+p_1\right)\left(s+p_2\right)=\left(s-\left(-1+j\right)\right)\left(s-\left(-1-j\right)\right)$$

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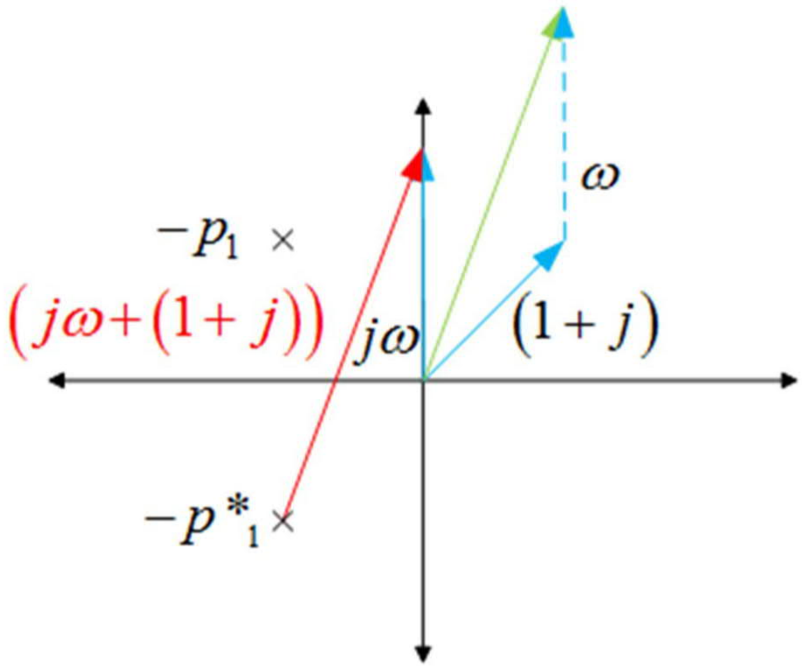
Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- Let's take a look for both terms
- Substitute *s* with $j\omega$ in $\left(s+\left(1+j\right)\right)$ and get $\left(j\omega+\left(1+j\right)\right)$ which is a complex number and can be drawn on the complex plane



- The green vector is the sum vector, and when it is moved to start at the second pole, its tip is located at the imaginary axis on the point ω.

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Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- Let's check

$$\left(s-\left(-1+j\right)\right)\left(s-\left(-1-j\right)\right)$$
$$=s^2-s\left(-1+j\right)-s\left(-1-j\right)+\left(-1+j\right)\left(-1-j\right)$$
$$=s^2+sj+s+sj+\left(-1+j\right)\left(-1-j\right)$$
$$=s^2+2s+1+1j-1j-jj=s^2+2s+2$$

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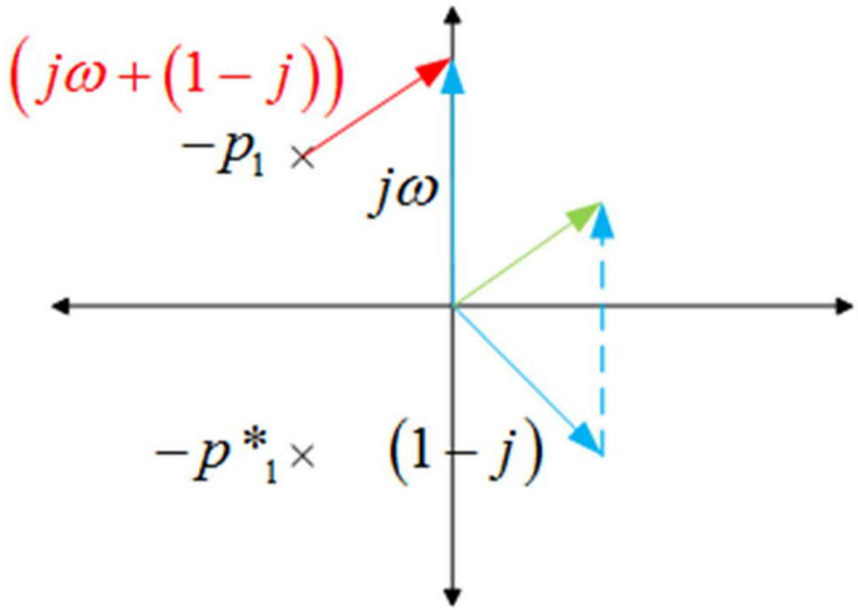
Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- Respectively, let's substitute *s* with $j\omega$ in $\left(s+\left(1-j\right)\right)$ and get $\left(j\omega+\left(1-j\right)\right)$ which is a complex number and can be drawn on the complex plane

- The green vector is the sum vector, and when it is moved to start at the second pole, its tip is located at the imaginary axis on the point ω.



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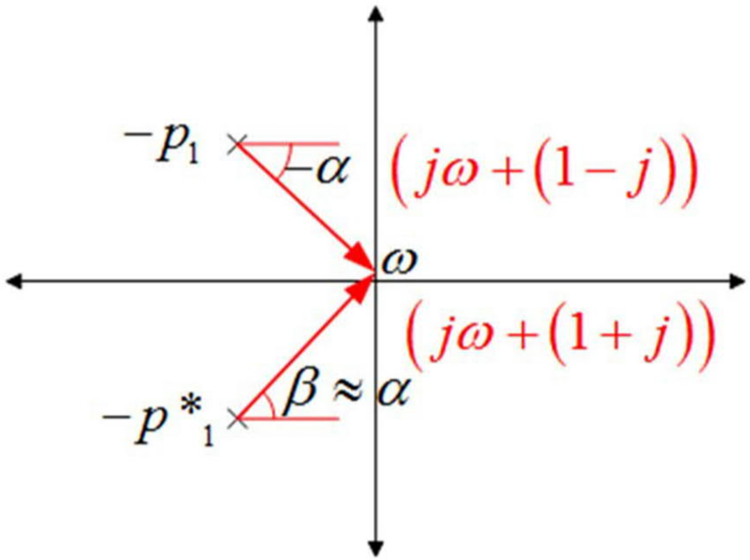
Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- Let's then analyze what is happening to gain and phase, when the frequency ω is changed.
- At very low frequencies (ω ≈ 0) the angles α and β are almost equal and their sum effect is near zero.



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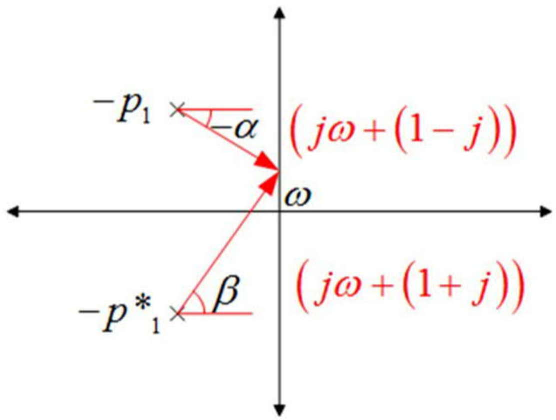
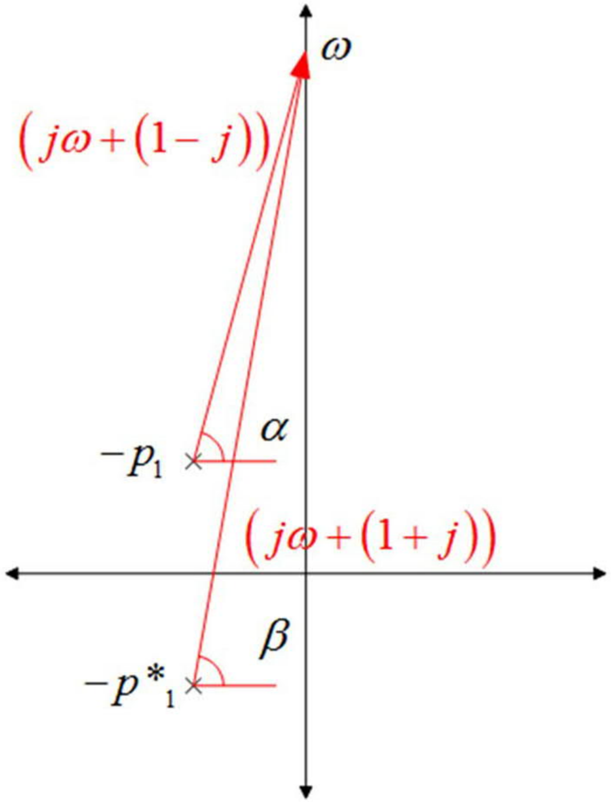
Transfer function in frequency domain

The role of the poles and the zeros to the frequency response

- With a little bit higher frequencies the angles are no more near equal. Their sum effect is a little positive. Because they are poles, their effect to the transfer function is the inverse of their absolute value and opposite number of the phase.



- With high frequencies, both vectors form near 90° angle with the real axis having a total effect of 180°.



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Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

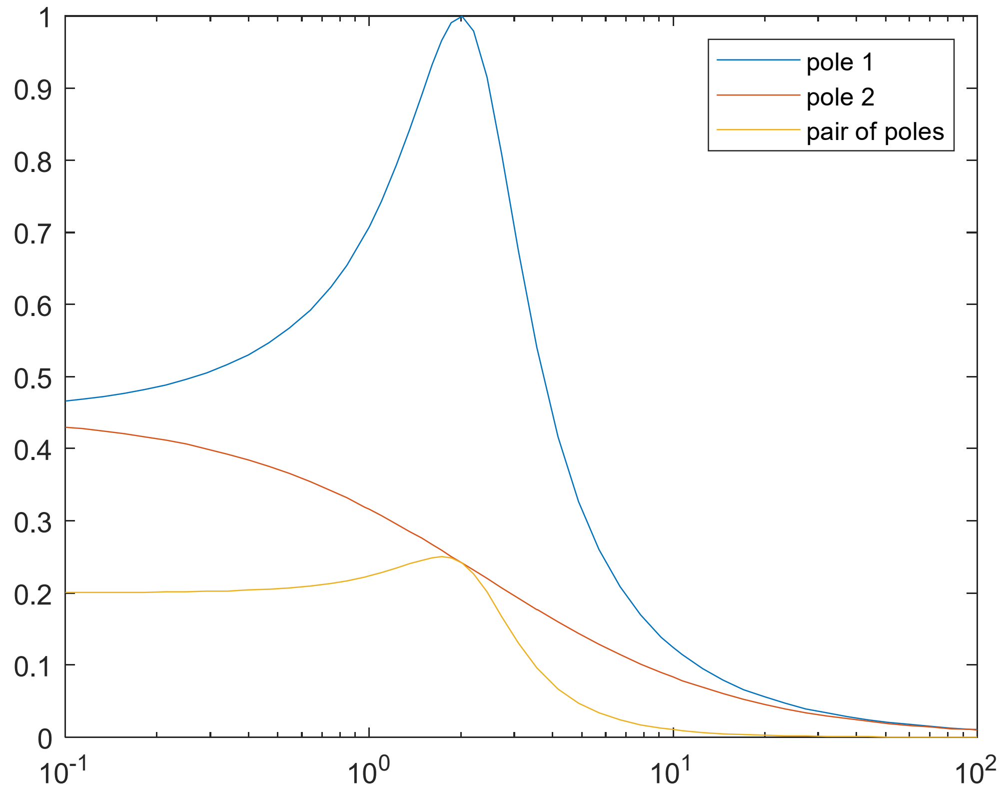
Matlab example

```
s=tf('s');
G=1/(s*s+2*s+5)
[mag,phase,wout] = bode(G);
rts= roots([1 2 5]);

figure(4)
g1=abs((wout*i-(rts(1))));
g2=abs((wout*i-(rts(2))));

semilogx(wout,[1./g1 1./g2 1./(g1.*g2)]);

legend('pole 1', 'pole 2', 'pair of poles')
```



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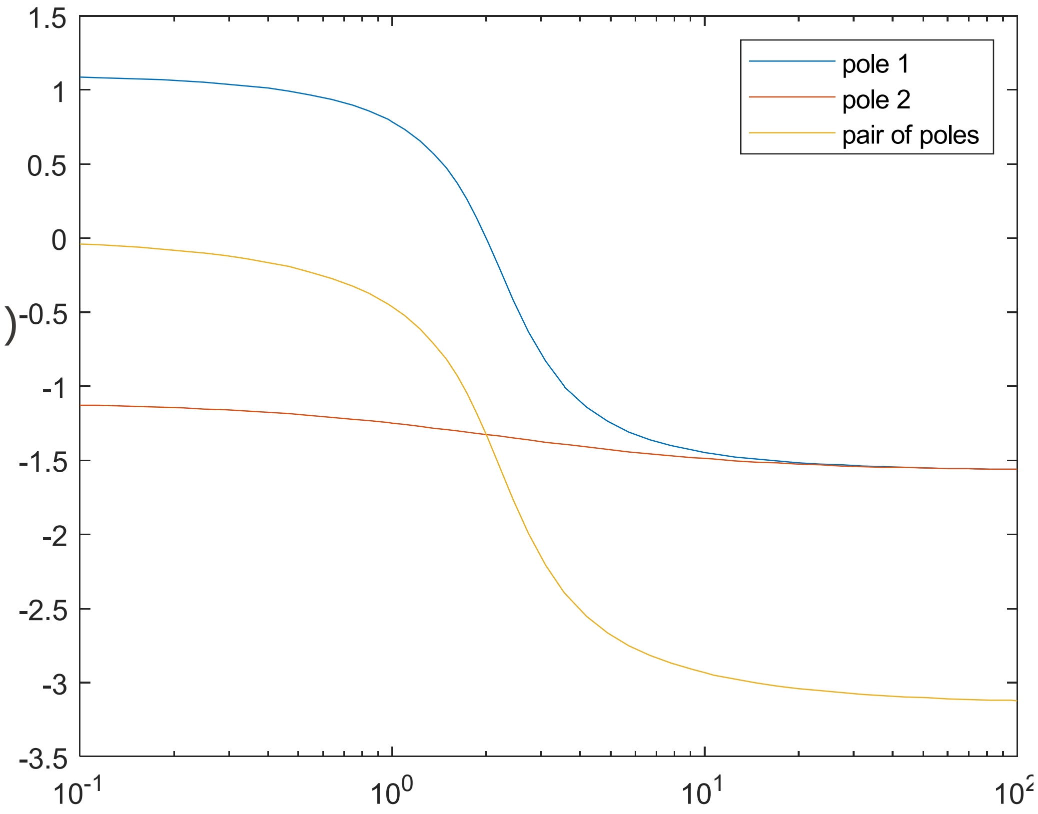
Transfer function in frequency domain

The role of the poles and the zeros to the frequency response

```
g1=angle((wout*i-(rts(1))));
g2=angle((wout*i-(rts(2))));

semilogx(wout,[-g1 -g2 -(g1+g2)]),

legend('pole 1', 'pole 2', 'pair of poles')
```



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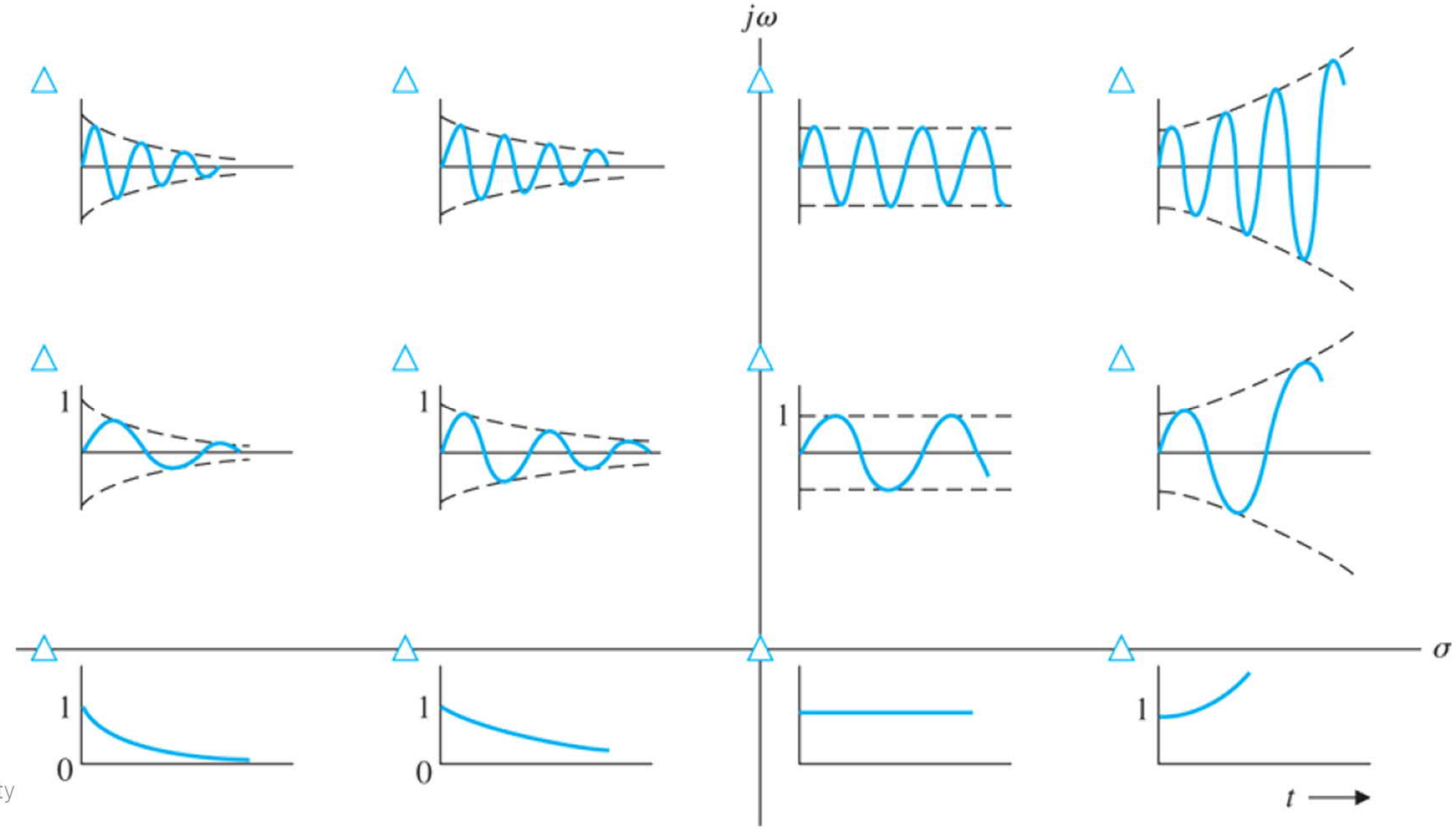
Transfer function in frequency domain
The role of the poles and the zeros to the frequency response

- Pole location on a complex plane vs. time response



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Stability in frequency domain

- Open loop – closed loop
- Characteristic polynome
- Mapping contours in the complex plane
- Gain and angle of the transfer function
- Cauchy’s theorem (Principle of the argument)
- Nyquist criterion for stability
- Relative stability
- Gain and phase margins from Nyquist diagram

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Stability in frequency domain

- The most important thing in dynamic system analysis is stability. If the system is stable, then the next question is the degree of stability
- A typical case is to make experiments with open system and according to results we define the feedback system having desired stability and performance properties
  - typically trade off between performance and stability
- From the open loop transfer function $G_{OL}(s)$ we get the closed loop system $G_{CL}(s)$ with negative feedback

$$G_{CL}(s)=\frac{G_{OL}(s)}{1+G_{OL}(s)}$$

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Stability in frequency domain

- The analysis of the closed system is made by inspection of the closed system poles, that is roots of the characteristic polynome (denominator of the transfer function).
- The characteristic equation is designated with F(s)

$$F(s) = 1 + G_{OL}(s)$$

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Stability in frequency domain
Mapping contours in the complex plane

- Some characteristics of the complex function F(s)
  - The argument *s* of the of the function F(s) is a complex number implying that also the value of F(s) is a complex number
  - E.g. F(s) = 2s + 1, and s = σ + jω

$$u + j\upsilon = F(s) = 2s + 1 = 2(\sigma + j\omega) + 1$$

  - F(s) maps complex number to another complex number so that

$u = 2\sigma + 1$ and $\upsilon = 2\omega$

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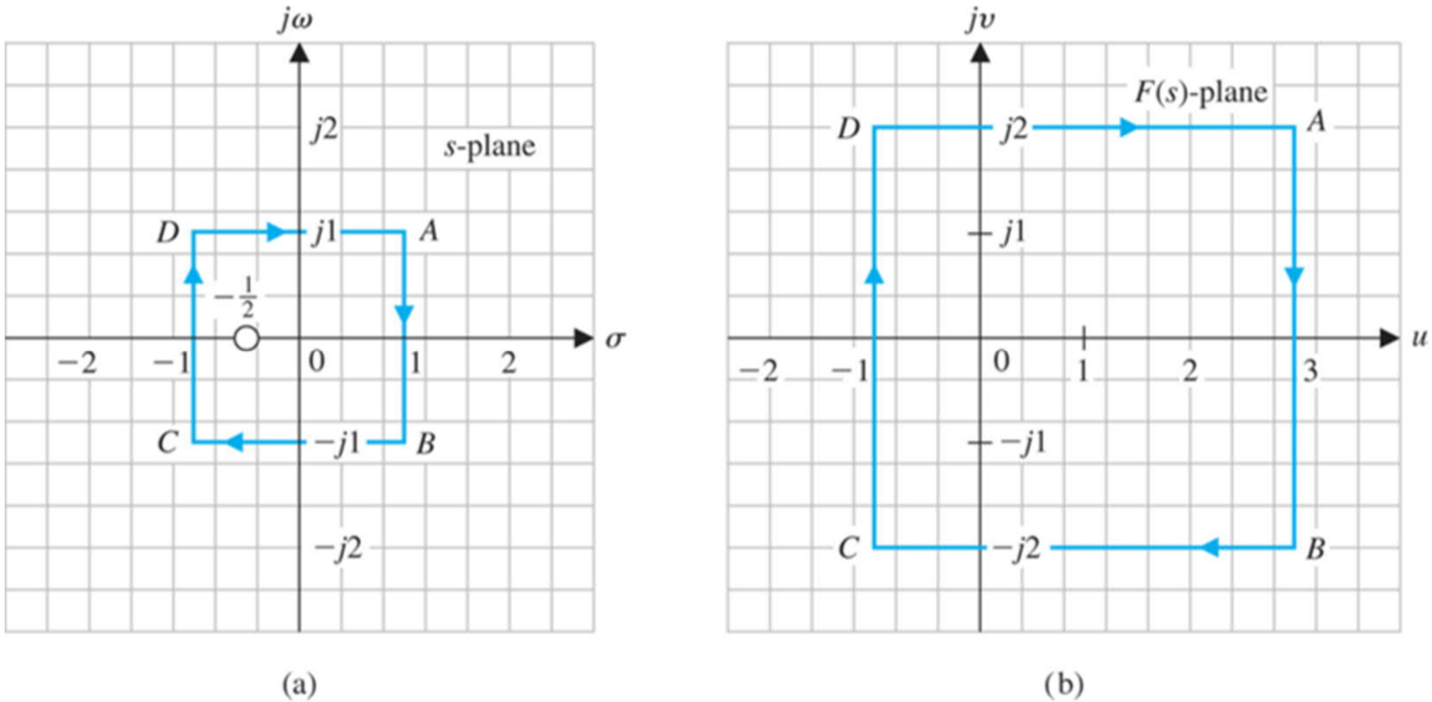
Stability in frequency domain
Mapping contours in the complex plane

- Thus the contour is mapped to a new contour, the center of which is shifted with one unit and the size is doubled



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Stability in frequency domain
Mapping contours in the complex plane

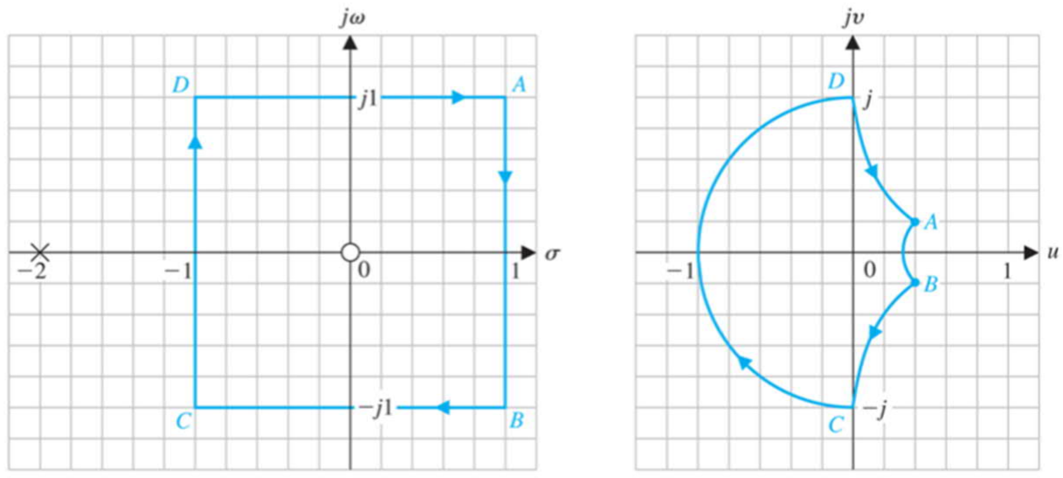
- Thus, the contour has been mapped by F(s) into a contour of an identical form, a square, with the center shifted by one unit and the magnitude of a side multiplied by two. This type of mapping, which retains the angles of the s-plane contour on the F(s)-plane, is called a **conformal mapping**.
- The points A, B, C, and D, as shown in the s-plane contour, map into the points A, B, C, and D shown in the F(s)-plane. Furthermore, a direction of traversal of the s-plane contour can be indicated by the direction ABCD and the arrows shown on the contour.
- Typically, we are concerned with an F(s) that is a rational function of s. Therefore, it will be worthwhile to consider another example of a mapping of a contour. Let us again consider the unit square contour for the function

$$F(s)=\frac{s}{s+2}$$

Stability in frequency domain
Mapping contours in the complex plane

- Several values of F(s) as s traverses the square contour are given in table below, and the resulting contour on the F(s)-plane is shown below.

	Point A		Point B		Point C		Point D	
s=σ+jω	$1 + j1$	1	$1 - j1$	$- j1$	$-1 - j1$	-1	$-1 + j1$	$j1$
F(s)= u+ju	$\frac{4+2j}{10}$	$\frac{1}{3}$	$\frac{4-2j}{10}$	$\frac{1-2j}{5}$	-j	-1	+j	$\frac{1-2j}{5}$



(a) (b)

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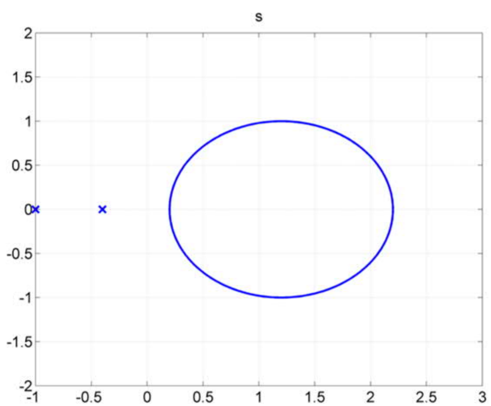
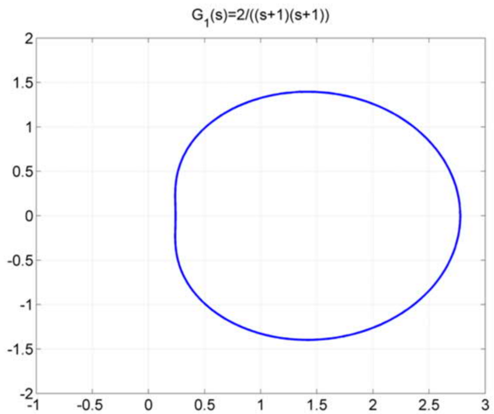
Stability in frequency domain
Mapping contours in the complex plane

Example

- Let's look at two examples, how transfer functions $G_1(s)$ and $G_2(s)$ map encirclements on s-plane to complex plane.
- Case 1: Transfer function $G_1(s) = \dfrac{1}{(s+1)(s+0.4)}$
- maps points of a circle, such that inside the circle there are no poles, to a complex plane. The poles of the $G_1(s)$ are s = -0.4 and s = -1. From the plot we can see that $G_1(s)$ maps the circle on the s-plane almost a circle on the complex plane.

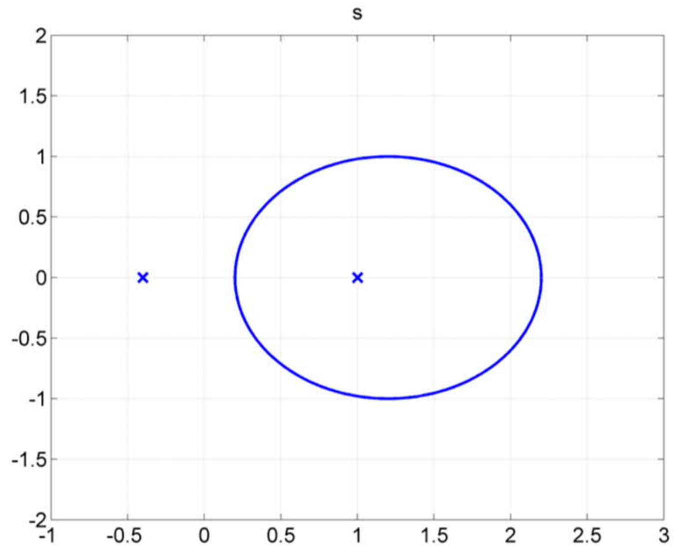


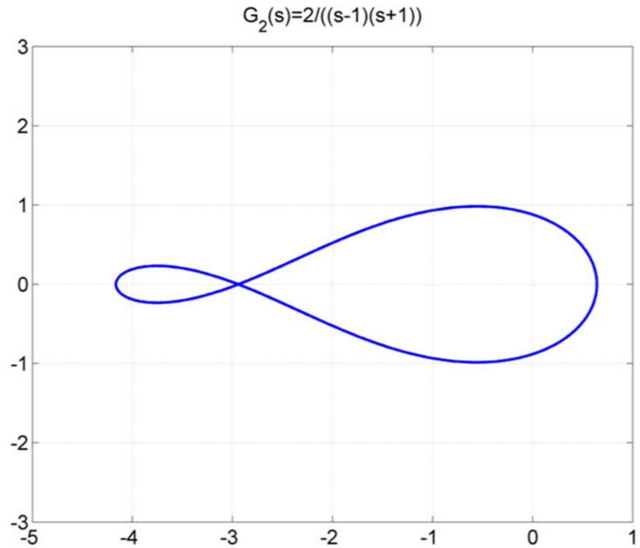


Stability in frequency domain
Mapping contours in the complex plane

- Case 2: Transfer function $G_2(s)=\dfrac{1}{(s-1)(s+0.4)}$

- The poles of $G_2(s)$ are s = -0.4 and s = 1, and the same circle as in case 1 in the s-plane encloses the pole s = 1. In this case the mapped route of $G_2(S)$ encircles the origin of the complex plane once.





Stability in frequency domain
Gain and angle of the transfer function

- Every transfer function has a property that it maps every complex number *s* (a vector on a complex plane) as a set of vectors starting from the zeros and poles to the value point of $s = (\sigma + j\omega)$. The total gain and angle of the transfer function is the sum of individual lengths (gains) and angles of these vectors.
- Let's look at the transfer function $G(s) = \frac{s+1}{(2+s)(3+s)}$
- The zeros and poles of the system are $z_1 = -1$; $p_1 = -2$; $p_2 = -3$.
- Let's substitute $s = 2 + j$.
- The value of G(s) can be calculated directly

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Stability in frequency domain
Gain and angle of the transfer function

$$G(s) = \frac{1+2+j}{(2+2+j)(3+2+j)} = \frac{3+j}{(4+j)(5+j)} = \frac{3+j}{20+9j-1} = \frac{(19-9j)(3+j)}{(19-9j)(19+9j)}$$
$$= \frac{57-8j+9}{361+81} = \frac{66-8j}{442}$$

$$|G(s)| = \sqrt{\left(\frac{66}{442}\right)^2 + \left(\frac{8}{442}\right)^2} \approx 0.1504$$

$$\angle G(s) = \operatorname{atan}\left(-\frac{8}{442} \Big/ \frac{66}{442}\right) \approx -0.1206$$

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Stability in frequency domain
Gain and angle of the transfer function

- With Matlab

```
>> s=2+i;
>> G=(1+s)/((2+s)*(3+s))
G =
   0.1493 - 0.0181i

>> abs(G)
ans =
    0.1504
>> angle(G)
ans =
   -0.1206
>>
```

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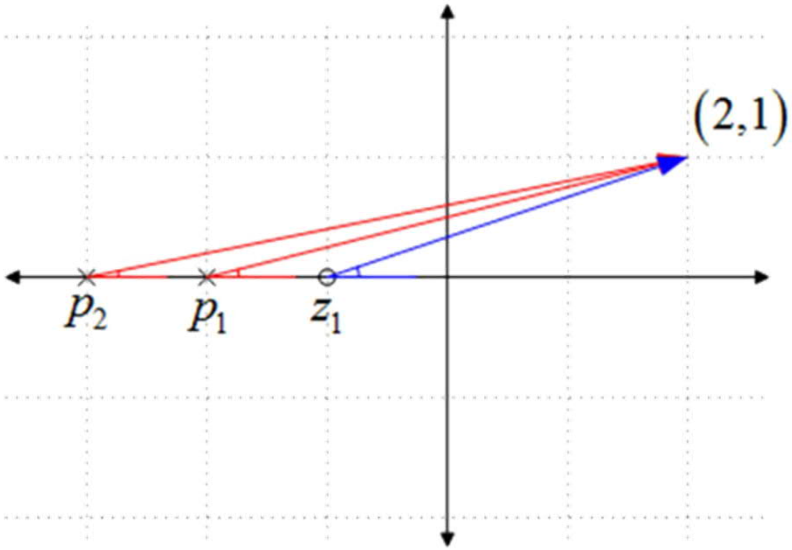
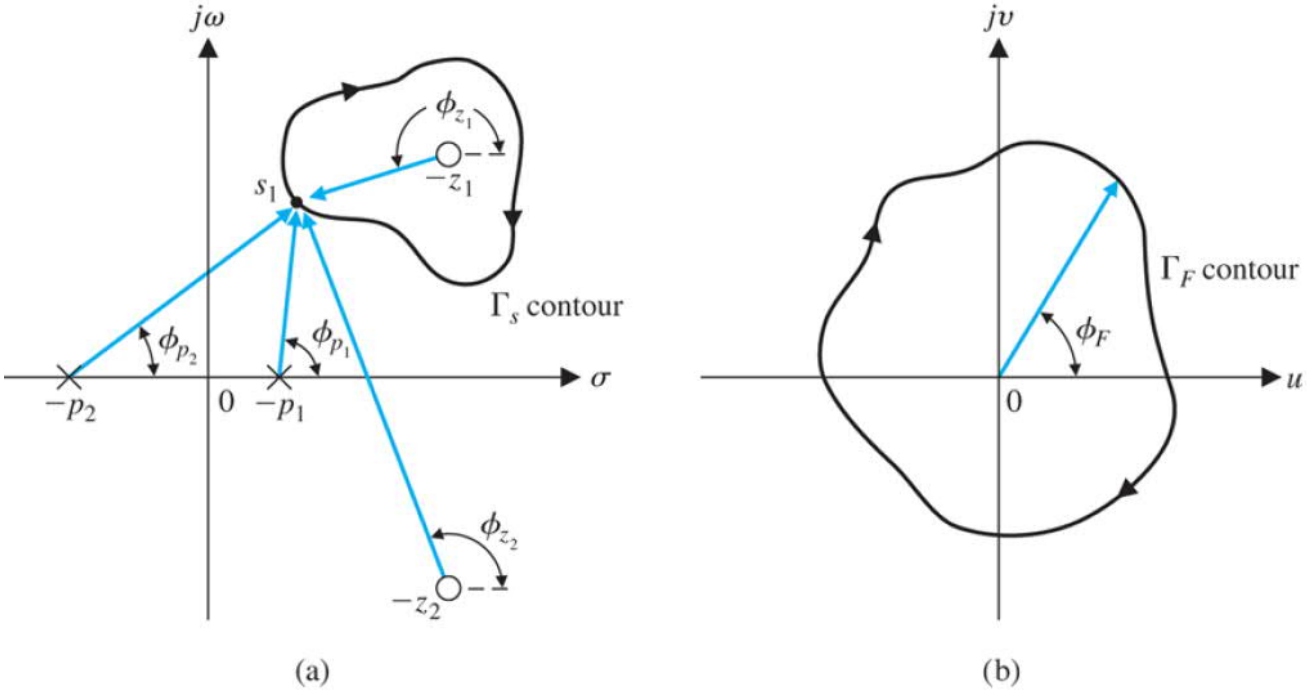
Stability in frequency domain
Gain and angle of the transfer function

$$|1+s| = |1+2+j| = \sqrt{3^2+1^2} = \sqrt{9+1} = \sqrt{10}$$
$$|2+s| = |2+2+j| = \sqrt{4^2+1^2} = \sqrt{16+1} = \sqrt{17}$$
$$|3+s| = |3+2+j| = \sqrt{5^2+1^2} = \sqrt{25+1} = \sqrt{26}$$
$$|G(s)| = \frac{\sqrt{10}}{\sqrt{17}\sqrt{26}} \approx 0.1504$$
$$\angle 1+s = \angle 1+2+j = \operatorname{atan}\left(\frac{1}{3}\right)$$
$$\angle 2+s = \angle 2+2+j = \operatorname{atan}\left(\frac{1}{4}\right)$$
$$\angle 3+s = \angle 3+2+j = \operatorname{atan}\left(\frac{1}{5}\right)$$
$$\angle G(s) = (\angle 1+2+j) - (\angle 2+2+j) - (\angle 3+2+j)$$
$$= \operatorname{atan}\left(\frac{1}{3}\right) - \operatorname{atan}\left(\frac{1}{4}\right) - \operatorname{atan}\left(\frac{1}{5}\right)$$
$$= -0.1206$$



Stability in frequency domain
Cauchy's theorem (Principle of the argument)

- If a contour Γ_S in the s-plane encircles Z zeros and P poles of F(s) and does not pass through any poles or zeros of F(s) and the traversal is in the clockwise direction along the contour, the corresponding contour Γ_F in the F(s)-plane encircles the origin of the F(s)-plane N = Z— P times in the clockwise direction.
  - clockwise encircling of zeros on s-plane results clockwise encircling of origin on F-plane
  - clockwise encircling of poles on s-plane results counterclockwise encircling of origin on F-plane



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Stability in frequency domain
Nyquist criterion for stability

- For the closed (controlled) system, we are looking for the characteristic equation of the system

$$F(s) = 1 + G_{OL}(s)$$

- To be stable, the zeros of F(s) must locate on the left half of the complex plane. Thus, the encircling area on the s-plane is selected to be the right half plane, and then we check if there is any any zeros on this half plane by utilizing Cauchy's theorem, Z = N + P.
- If P = 0, the usual case for characteristic equation, the number of unstable poles of the closed (controlled) system is equal with the number of circles of F(s) around the origin of the F-plane.

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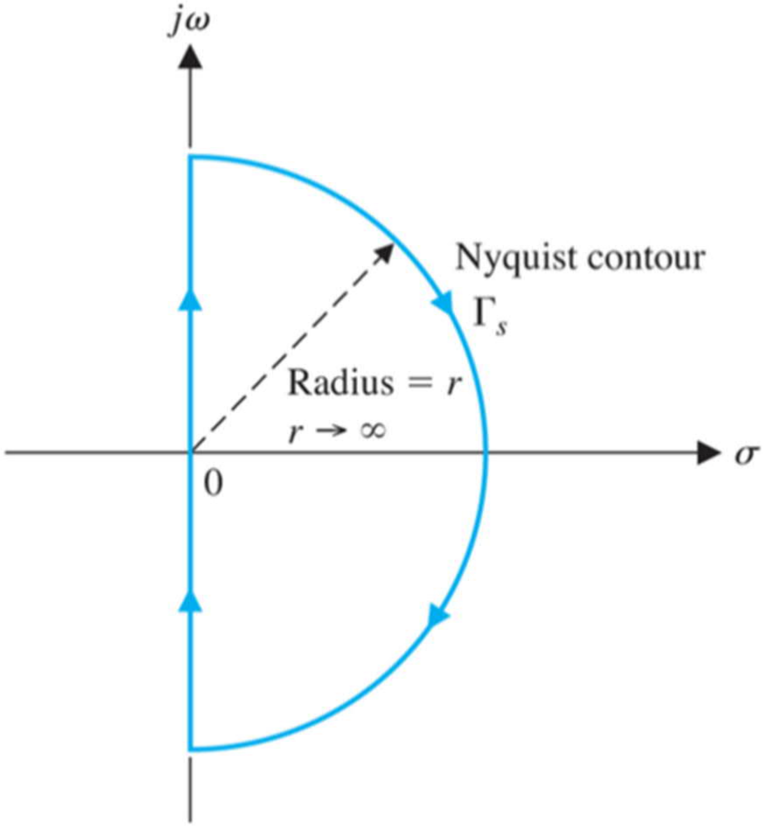
Stability in frequency domain
Nyquist criterion for stability

- Nyquist contour closing the whole right half of the complex plane
- along imaginary axis from -j∞ to +j∞ connected with half circle with the infinite radius



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Stability in frequency domain
Nyquist criterion for stability

- Instead of $F(s) = 1 + G_{OL}(s)$ we can alternatively inspect the function $F'(s) = F(s) - 1 = G_{OL}(s)$.
- This is convenient, because $G_{OL}(s)$ is typically factorized to zeros and poles

$$G_{OL}(s) = K\frac{(z_1+s)\cdots(z_n+s)}{(p_1+s)\cdots(p_m+s)}$$

- When using F'(s) instead of F(s), the circles are not counted around the origin but around the point (-1 *j*0)

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Stability in frequency domain
Nyquist criterion for stability

- A feedback system is stable if and only if the contour Γ_L in the $G_{OL}(s)$-plane does not encircle the (-1, 0) point when the number of poles of $G_{OL}(s)$ in the right-hand s-plane is zero (P = 0).
- When the number of poles of $G_{OL}(s)$ in the right-hand s-plane is other than zero: the Nyquist criterion is stated as follows:
- A feedback control system is stable if and only if, for the contour Γ_L the number of counterclockwise encirclements of the (-1, 0) point is equal to the number of poles of $G_{OL}(s)$ with positive real parts.

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Stability in frequency domain
Relative stability

- The Nyquist stability criterion is defined in terms of the (-1,0) point on the polar plot or the 0 dB, -180° point on the Bode diagram or log-magnitude-phase diagram.
- Clearly, the proximity of the $G_{OL}(j\omega)$-locus to this stability point is a measure of the relative stability of a system.
- Let's take a look at the frequency response of the following system with different values of gain K.

$$G_{OL}(s) = \frac{K}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

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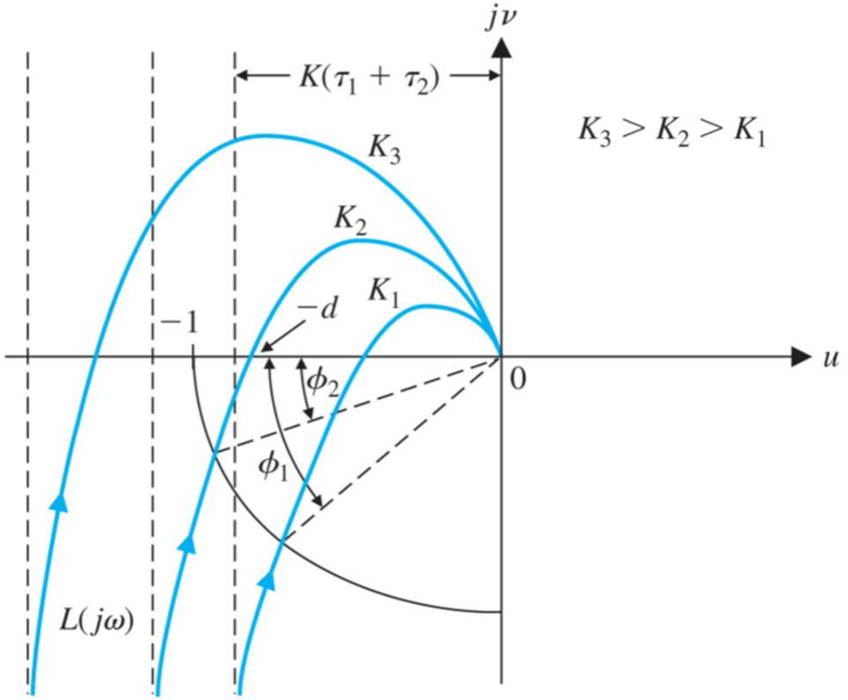
Stability in frequency domain
Relative stability

- As K increases, the polar plot approaches the -1 point and eventually encircles the -1 point for a gain $K = K_3$. The locus intersects the u-axis at a point

$$u = \frac{-K\tau_1\tau_2}{\tau_1+\tau_2}$$



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Stability in frequency domain
Gain and phase margins from Nyquist diagram

Gain margin

- The system has roots on the jω-axis when

$$u = -1 \quad \text{or} \quad K = \frac{\tau_1 + \tau_2}{\tau_1 \tau_2}$$

- As K is decreased below this marginal value, the stability is increased, and the margin between the critical gain $K = (\tau_1 + \tau_2)/\tau_1\tau_2$ and a gain $K = K_2$ is a measure of the relative stability.
- This measure of relative stability is called the gain margin and is defined as the reciprocal of the gain $|G_{OL}(j\omega)|$ at the frequency at which the phase angle reaches — 180° (that is, $\nu = 0$).
- The gain margin is a measure of the factor by which the system gain would have to be increased for the $G_{OL}(j\omega)$ locus to pass through the u = -1 point. Thus, for a gain $K = K_2$ in figure, the gain margin is equal to the reciprocal of $G_{OL}(j\omega)$ when $\nu = 0$, that is *1/d*.

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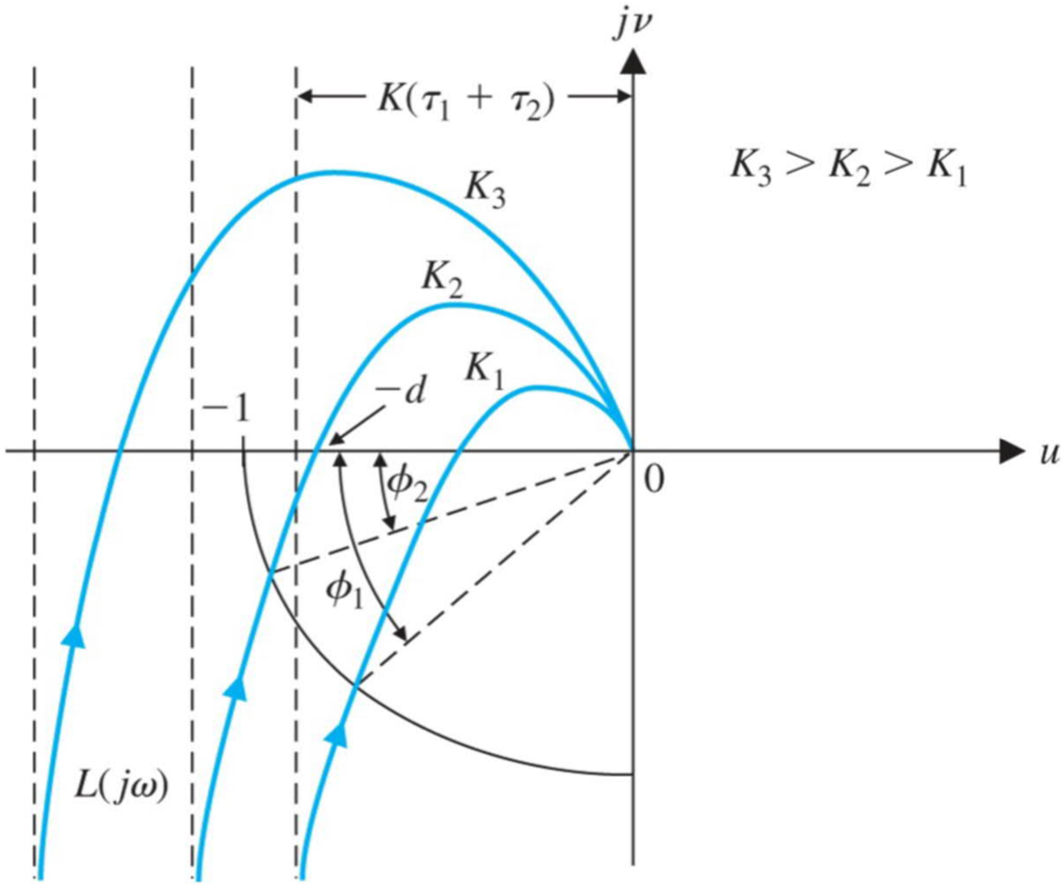
Stability in frequency domain
Gain and phase margins from Nyquist diagram

- The gain margin is the increase in the system gain when phase = -180° that will result in a marginally stable system with intersection of the -1 + j0 point on the Nyquist diagram.



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Stability in frequency domain
Gain and phase margins from Nyquist diagram

Phase margin

- An alternative measure of relative stability can be defined in terms of the phase angle margin between a specific system and a system that is marginally stable.
- The phase margin is defined as the phase angle through which the $G_{OL}(j\omega)$ locus must be rotated so that the unity magnitude $|G_{OL}(j\omega)| = 1$ point will pass through the (-1, 0) point in the $G_{OL}(j\omega)$ plane.
- This measure of relative stability is equal to the additional phase lag required before the system becomes unstable.

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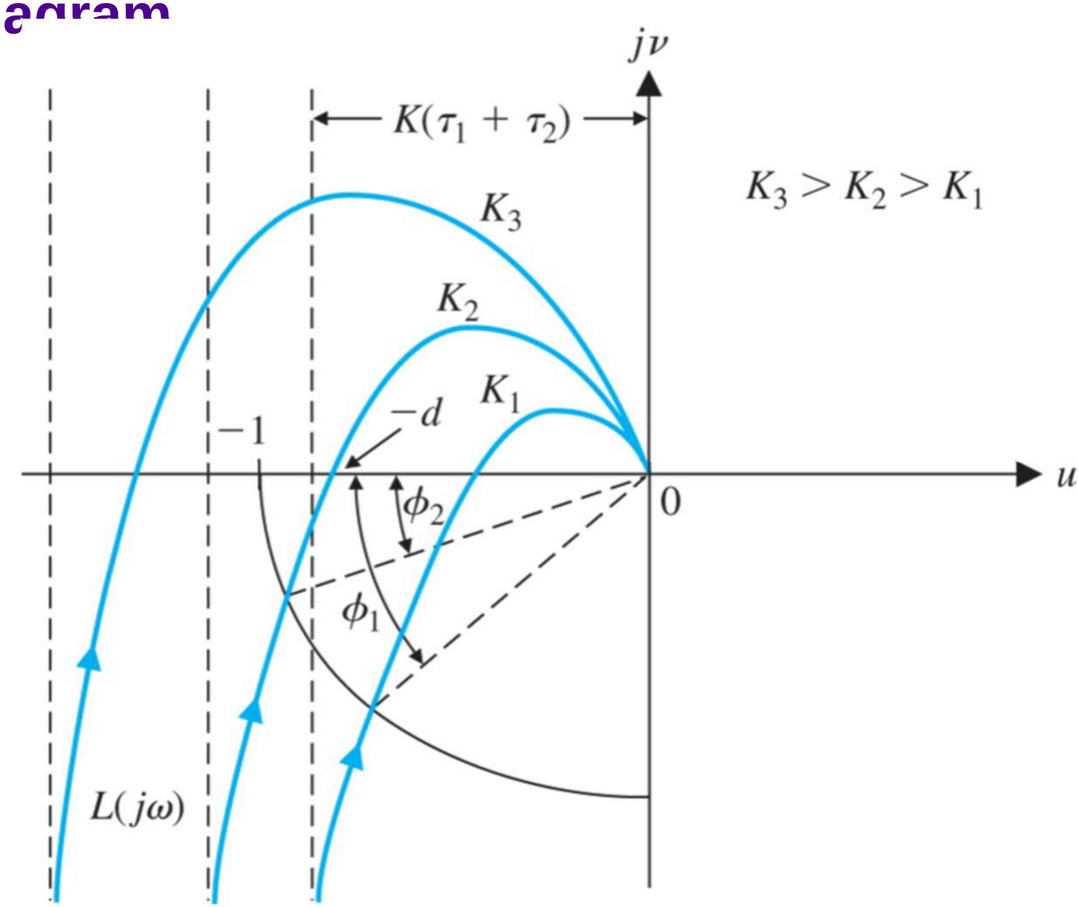
Stability in frequency domain

Gain and phase margins from Nyquist diagram

- For a gain $K = K_2$, an additional phase angle, ϕ_2, may be added to the system before the system becomes unstable. Similarly, for the gain K_1, the phase margin is equal to ϕ_1.
- The phase margin is the amount of phase shift of the $G_{OL}(j\omega)$ at unity magnitude that will result in a marginally stable system with intersection of the $-1 + j0$ point on the Nyquist diagram.



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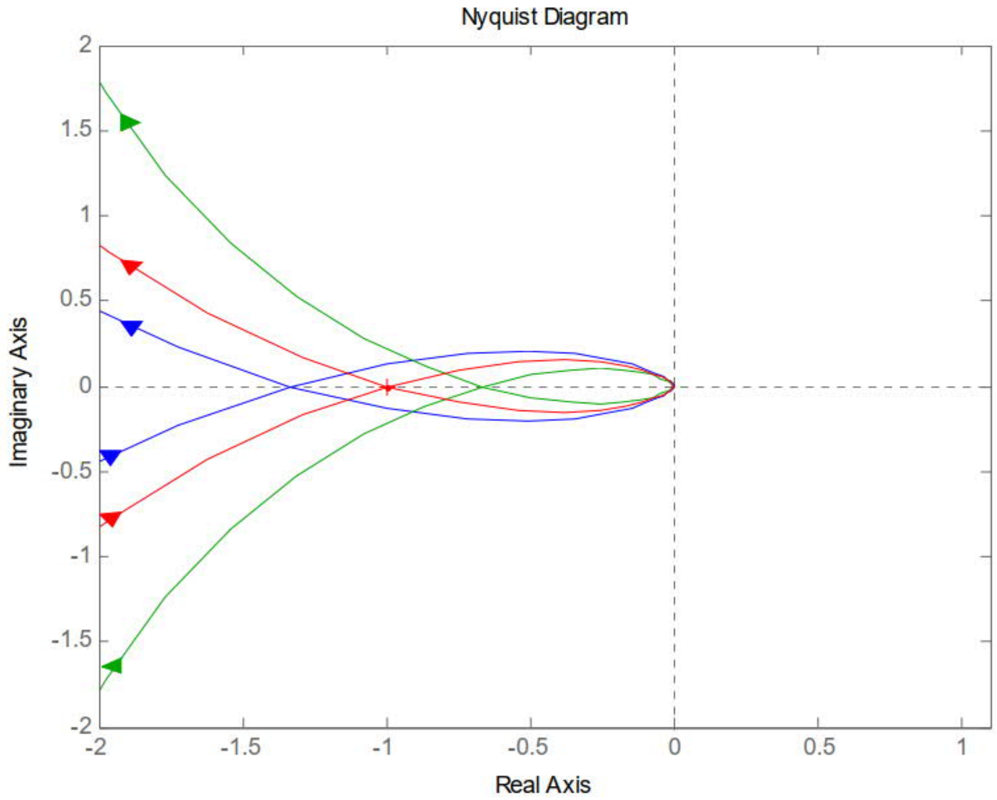
Stability in frequency domain
Gain and phase margins from Nyquist diagram

- Example: A basic feedback system

$$G_{OL}(s)=\frac{K}{s(\tau_1 s+1)(\tau_2 s+1)}$$

- With Matlab:
- `s=tf('s')`
- `K=0.5; tau1=4; tau2=2;`
- `G=K/(s*(tau1*s+1)*(tau2*s+1))`
- `figure(1),`
- `nyquist(G),`
- `In fig`
- In the figure, K = 0.5, 0.75 and 1.0



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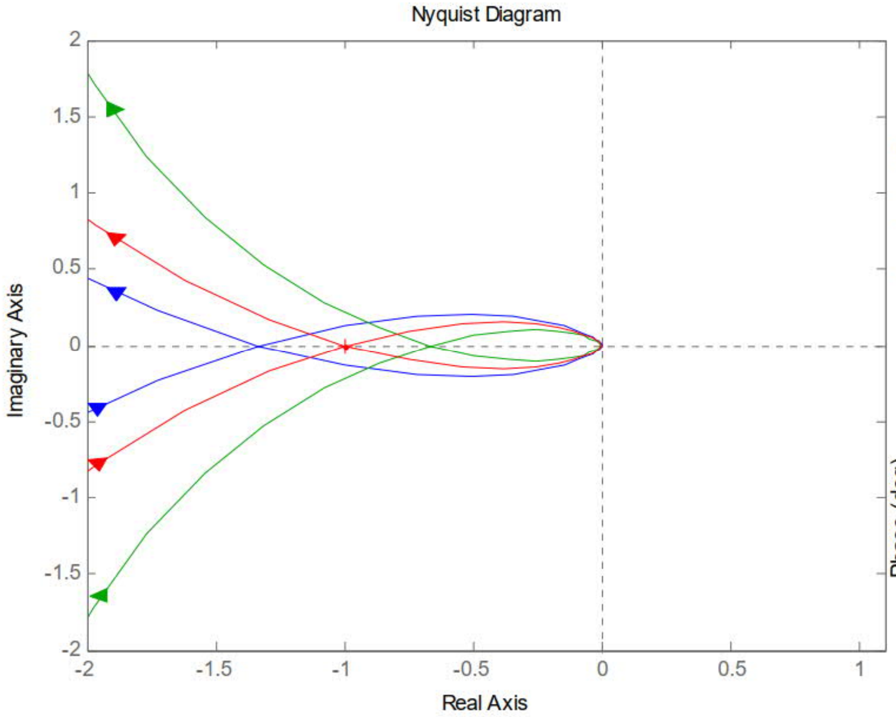
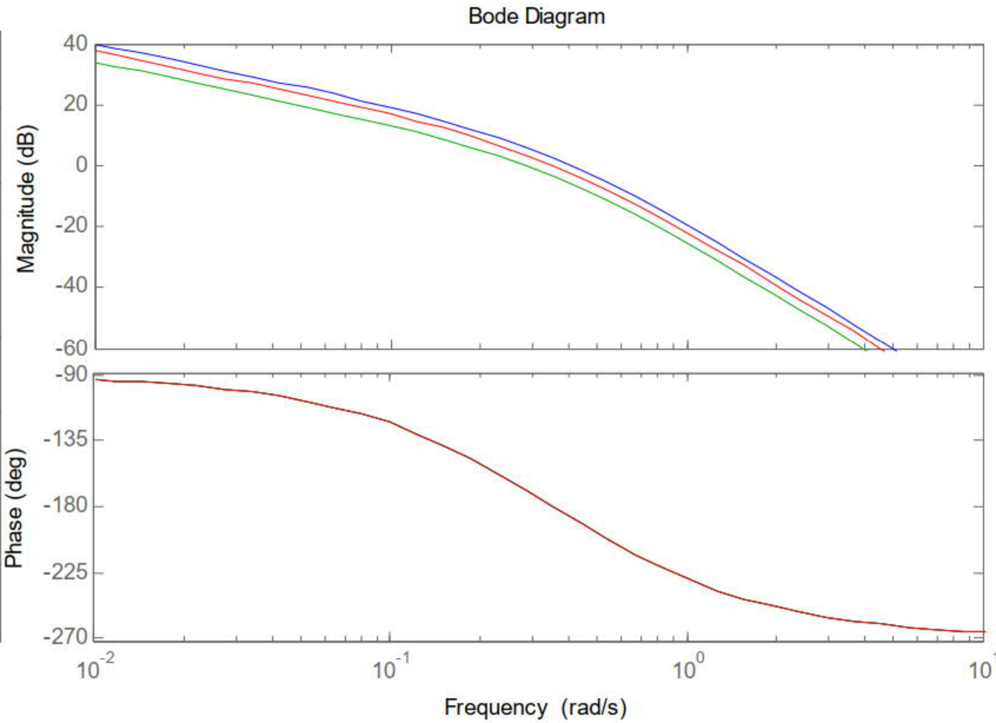
Stability in frequency domain
Gain and phase margins from Nyquist diagram





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Stability in frequency domain
Gain and phase margins from Nyquist diagram

- Effect of system delay to stability
- Transfer function for pure delay T is

$$G_d(s) = e^{-sT}$$

- The delay causes phase delay for the system but has no effect on the system gain

$$\phi(\omega) = -\omega T$$

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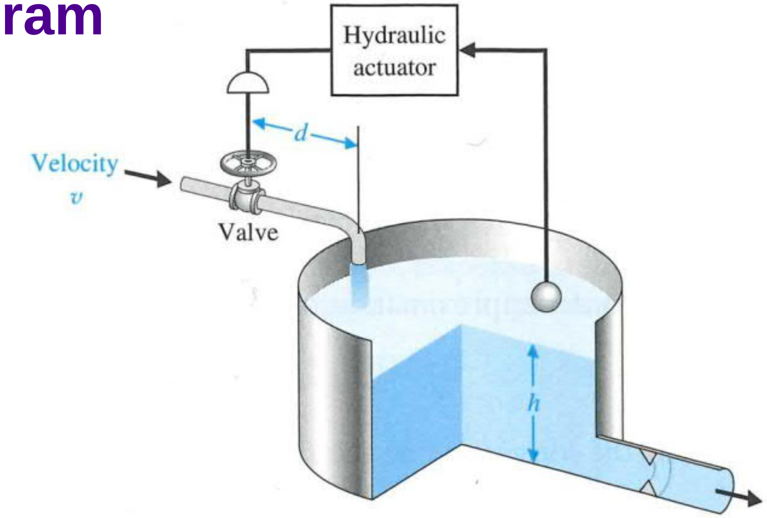
Stability in frequency domain
Gain and phase margins from Nyquist diagram

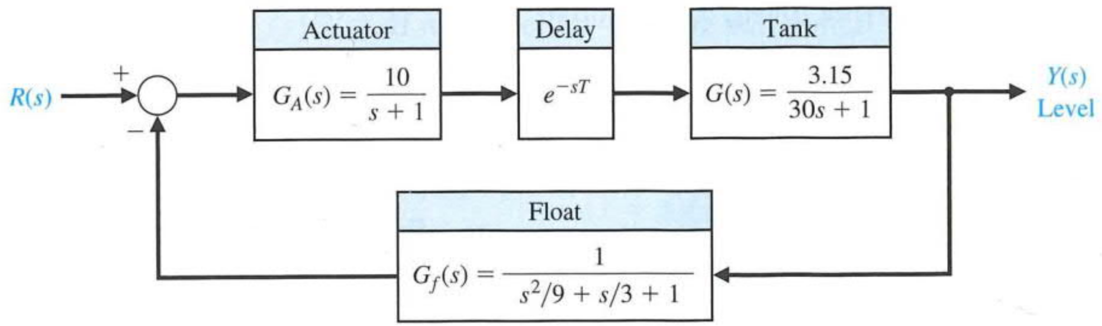
Example: Level control

- The system model is

$$GH(s) = G_A G(s) G_f(s) e^{-sT}$$
$$= \frac{31.5}{(s+1)(30s+1)\left((s^2/9 + s/3 + 1)\right)} e^{-sT}$$



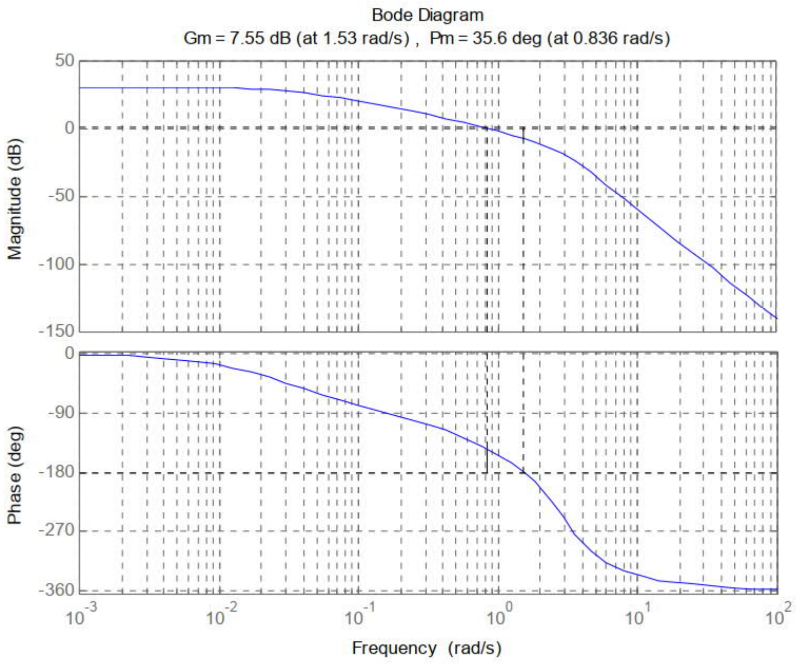
(a)

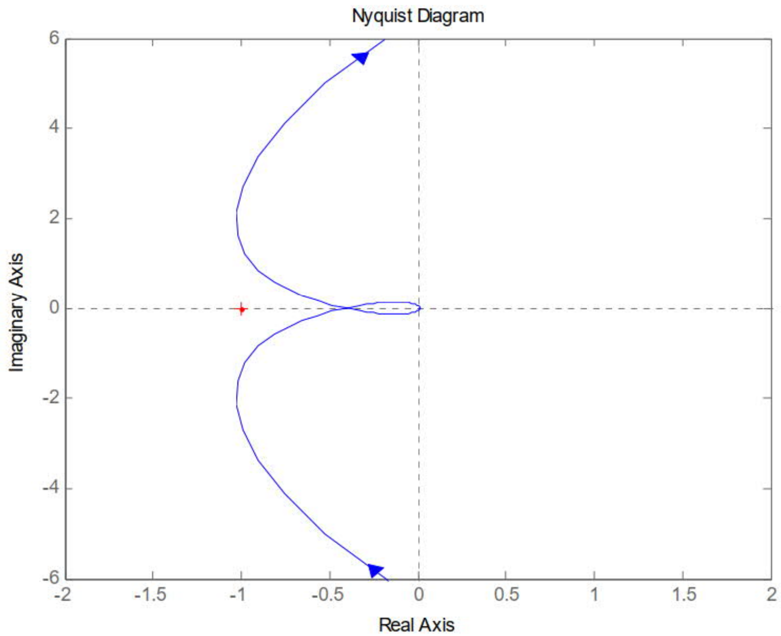


(b)

Stability in frequency domain
Gain and phase margins from Nyquist diagram

- Bode and Nyquist diagrams of the non-delayed system





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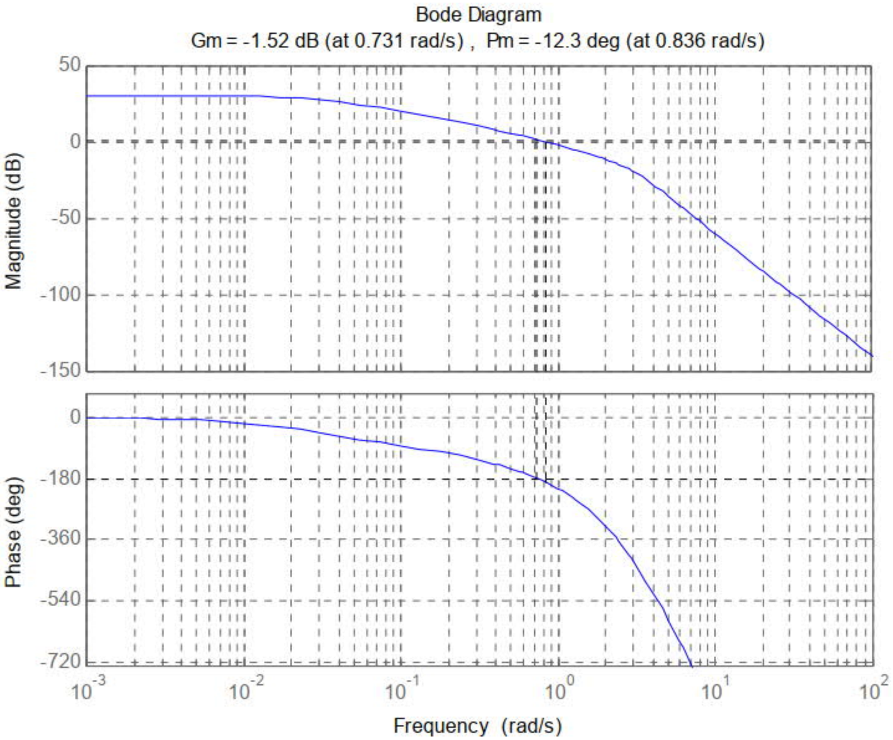
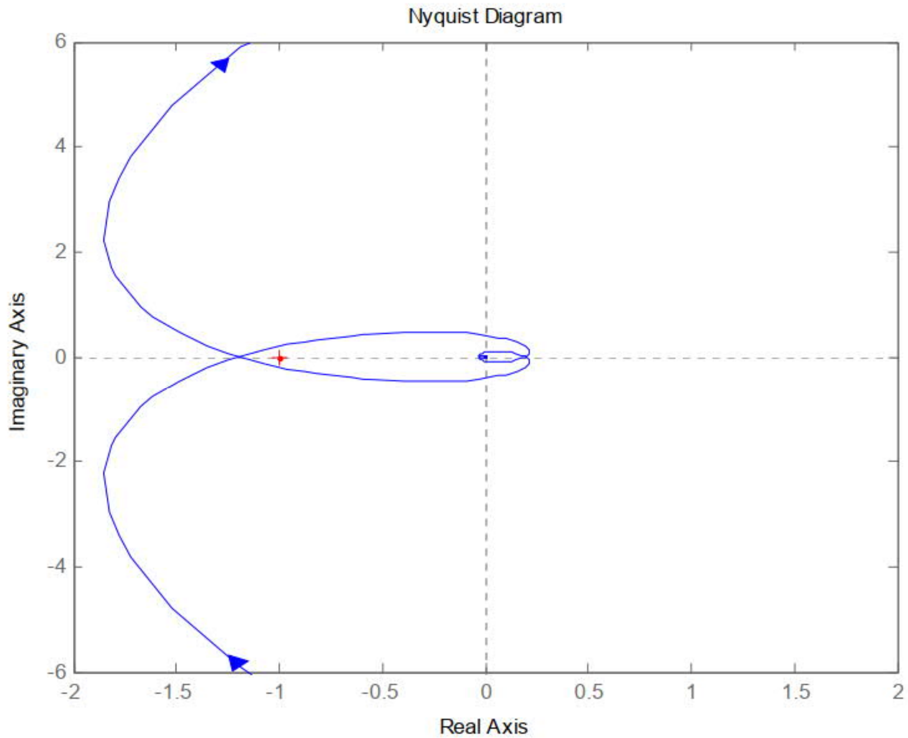
Stability in frequency domain
Gain and phase margins from Nyquist diagram

- Bode and Nyquist diagrams of the delayed system





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Stability in frequency domain
Gain and phase margins from Nyquist diagram

- Bode diagrams of non-delayed and delayed systems